CURRENCY TRADING

Tuesday, May 15, 2001

EXCHANGE RATES

The New York foreign exchange mid-range rates below apply to trading among banks in amounts of $1 million and more, as quoted at 4 p.m. Eastern time by Reuters and other sources. Retail transactions provide fewer units of foreign currency per dollar. Rates for the 12 Euro currency countries are derived from the latest dollar-euro rate using the exchange ratios set 1/1/99.

	U.S. $ EQUIV.		CURRENCY PER U.S. $	
Country	Tue	Mon	Tue	Mon
Argentina (Peso)	1.0003	1.0004	.9997	.9996
Australia (Dollar)	.5216	.5198	1.9170	1.9240
Austria (Schilling)	.06386	.06353	15.659	15.740
Bahrain (Dinar)	2.6525	2.6525	.3770	.3770
Belgium (Franc)	.0218	.0217	45.9060	46.1423
Brazil (Real)	.4274	.4322	2.3396	2.3140
Britain (Pound)	1.4241	1.4201	.7022	.7042
1-month forward	1.4227	1.4187	.7029	.7049
3-months forward	1.4201	1.4163	.7042	.7061
6-months forward	1.4164	1.4130	.7060	.7077
Canada (Dollar)	.6466	.6447	1.5465	1.5510
1-month forward	.6464	.6445	1.5471	1.5516
3-months forward	.6460	.6441	1.5480	1.5525
6-months forward	.6455	.6437	1.5492	1.5536
Chile (Peso)	.001650	.001654	605.95	604.45
China (Renminbi)	.1208	.1208	8.2769	8.2771
Colombia (Peso)	.0004206	.0004207	2377.50	2376.75
Czech. Rep. (Koruna)				
Commercial rate	.02552	.02542	39.182	39.332
Denmark (Krone)	.1177	.1172	8.4930	8.5346
Ecuador (US Dollar)-e	1.0000	1.0000	1.0000	1.0000
Finland (Markka)	.1478	.1470	6.7661	6.8009
France (Franc)	.1340	.1333	7.4647	7.5031
1-month forward	.1339	.1332	7.4681	7.5062
3-months forward	.1338	.1331	7.4753	7.5120
6-months forward	.1336	.1330	7.4829	7.5170
Germany (Mark)	.4493	.4470	2.2257	2.2372
1-month forward	.4491	.4468	2.2267	2.2381
3-months forward	.4487	.4465	2.2288	2.2398
6-months forward	.4482	.4462	2.2311	2.2413
Greece (Drachma)	.002579	.002566	387.77	389.76
Hong Kong (Dollar)	.1282	.1282	7.7999	7.7998
Hungary (Forint)	.003406	.003400	293.62	294.10
India (Rupee)	.02128	.02119	47.000	47.190
Indonesia (Rupiah)	.0000874	.0000877	11445	11400
Ireland (Punt)	1.1158	1.1101	.8962	.9008
Israel (Shekel)	.2411	.2410	4.1480	4.1500
Italy (Lira)	.0004538	.0004515	2203.44	2214.78
Japan (Yen)	.008098	.008114	123.48	123.24
1-month forward	.008127	.008144	123.04	122.78
3-months forward	.008180	.008197	122.25	121.99
6-months forward	.008261	.008281	121.05	120.77
Jordan (Dinar)	1.4069	1.4069	.7108	.7108
Kuwait (Dinar)	3.2478	3.2520	.3079	.3075
Lebanon (Pound)	.0006604	.0006604	1514.25	1514.25
Malaysia (Ringgit)-b	.2632	.2632	3.8001	3.8000
Malta (Lira)	2.2002	2.1930	.4545	.4560
Mexico (Peso)				
Floating rate	.1088	.1092	9.1890	9.1550
Netherlands (Guilder)	.3988	.3967	2.5078	2.5207
New Zealand (Dollar)	.4191	.4205	2.3861	2.3781
Norway (Krone)	.1095	.1092	9.1329	9.1583
Pakistan (Rupee)	.01623	.01627	61.600	61.450
Peru (new Sol)	.2779	.2776	3.5980	3.6025
Philippines (Peso)	.01992	.01988	50.200	50.300
Poland (Zloty)-d	.2497	.2491	4.0050	4.0150
Portugal (Escudo)	.004383	.004361	228.14	229.32
Russia (Ruble)-a	.03443	.03442	29.043	29.053
Saudi Arabia (Riyal)	.2666	.2666	3.7506	3.7506
Singapore (Dollar)	.5498	.5493	1.8190	1.8205
Slovak Rep. (Koruna)	.02036	.02026	49.121	49.352
South Africa (Rand)	.1252	.1256	7.9850	7.9625
South Korea (Won)	.0007673	.0007708	1303.35	1297.30
Spain (Peseta)	.005281	.005254	189.34	190.32
Sweden (Krona)	.0975	.0971	10.2611	10.2970
Switzerland (Franc)	.5734	.5710	1.7439	1.7513
1-month forward	.5739	.5715	1.7426	1.7498
3-months forward	.5747	.5724	1.7400	1.7470
6-months forward	.5761	.5740	1.7358	1.7421
Taiwan (Dollar)	.03040	.03041	32.900	32.880
Thailand (Baht)	.02197	.02196	45.525	45.536
Turkey (Lira)-f	.00000087	.00000087	1145000	1155000
United Arab (Dirham)	.2723	.2723	3.6730	3.6730
Uruguay (New Peso)				
Financial	.07648	.07663	13.075	13.050
Venezuela (Bolivar)	.001400	.001400	714.25	714.13
SDR	1.2602	1.2599	.7935	.7937
Euro	.8788	.8743	1.1379	1.1438

Special Drawing Rights (SDR) are based on exchange rates for the U.S., German, British, French, and Japanese currencies. Source: International Monetary Fund.

a-Russian Central Bank rate. b-Government rate. d-Floating rate; trading band suspended on 4/11/00. e-Adopted U.S. dollar as of 9/11/00. f-Floating rate, eff. Feb. 22.

PHILADELPHIA EXCHANGE OPTIONS

		CALLS VOL.	CALLS LAST	PUT VOL.	PUT LAST
ADlr					54.26
50,000 Australian Dollar EOM-European.					
52	May	6	0.30	–	–
CDollr					65.48
50,000 Canadian Dollars-cents per unit.					
65	Sep	–	–	100	1.08
JYen					92.53
6,250,000J.Yen-100ths of a cent per unit.					
79	Sep	–	–	2	0.92
80	Sep	–	–	6	1.18
81	Jun	–	–	2	0.86
81	Jul	–	–	40	1.12
81	Sep	–	–	15	1.50
83	Sep	1	1.55	–	–
Euro					88.15
62,500 Euro-European style					
88	May	5	0.47	–	–
90	May	8	0.15	–	–
Euro					88.15
62,500 Euro-European style.					
94	Sep	10	0.63	–	–
Euro					88.15
62,500 Euro-cents per unit.					
88	Jun	1	1.37	–	–
90	Sep	4	1.54	–	–
92	Jun	1	0.15	–	–
92	Sep	1	1.00	–	–
SFranc					57.83
62,500 Swiss Francs-cents per unit.					
59	Jun	8	0.19	–	–

Call Vol 46 Open Int 12,803
Put Vol 166 Open Int 19,561

KEY CURRENCY CROSS RATES

Late New York Trading Tuesday, May 15, 2001

	Dollar	Euro	Pound	SFranc	Guilder	Peso	Yen	Lira	D-Mark	FFranc	CdnDlr
Canada	1.5465	1.3591	2.2024	0.8868	.61668	.16830	.01252	.00070	.69484	.20718	....
France	7.4647	6.5600	10.6305	4.2805	2.9766	.81235	.06045	.00339	3.3539	....	4.8268
Germany	2.2257	1.9559	3.1696	1.2763	.88751	.24221	.01802	.00101	....	.29816	1.4392
Italy	2203.4	1936.4	3137.9	1263.5	878.63	239.79	17.844	....	989.99	295.18	1424.8
Japan	123.48	108.51	175.85	70.807	49.238	13.438	....	.05604	55.479	16.542	79.845
Mexico	9.1890	8.0753	13.086	5.2692	3.6642	....	.07442	.00417	4.1286	1.2310	5.9418
Netherlands	2.5078	2.2039	3.5714	1.4380	....	.27291	.02031	.00114	1.1267	.33595	1.6216
Switzerland	1.7439	1.5325	2.4835	....	.69539	.18978	.01412	.00079	.78353	.23362	1.1276
U.K.	.70220	.6171	....	.4027	.28001	.07642	.00569	.00032	.31550	.09407	.45406
Euro	1.13790	....	1.6205	.65251	.45375	.12383	.00922	.00052	.51126	.15244	.73580
U.S.	....	.8788	1.4241	.57343	.39876	.10883	.00810	.00045	.44930	.13396	.64662

Source: Reuters

Figure 14.7 Currency option prices from *The Wall Street Journal*. *Source:* (Reprinted with permission of *The Wall Street Journal*, p. C16. © May 16, 2001.)

14.15 FUTURES OPTIONS

A call futures option gives the owner the right, but not the obligation, to assume a long position in an underlying futures contract. As in other options, the buyer of a futures call option pays a premium to the futures call option seller (writer). The writer of a futures call has the obligation to deliver a futures contract to the futures call owner, should the futures call owner decide to exercise the option. If the writer had written a naked futures option, upon exercise he will have a short position in a futures contract. If the option were written as part of a covered write, the writer will just deliver the futures contract he owns. That is, the call owner will effectively assume the long futures position at the strike price of the futures option. Upon exercise of a futures call, the positions are marked to market. Thus, whereas the exerciser of an "ordinary" call must pay $\$K$ to acquire the asset, the exerciser of a futures call is actually paid the option's intrinsic value, $F-K$, to go long a futures contract. Note that F is the futures settlement price on the exercise day.

The owner of a put option on a futures contract has the right to go short a futures contract at the strike price. The writer of a futures put must accept a long position in a futures contract, should he ever be assigned an exercise. Again, all positions are marked to market once the owner has exercised his put.

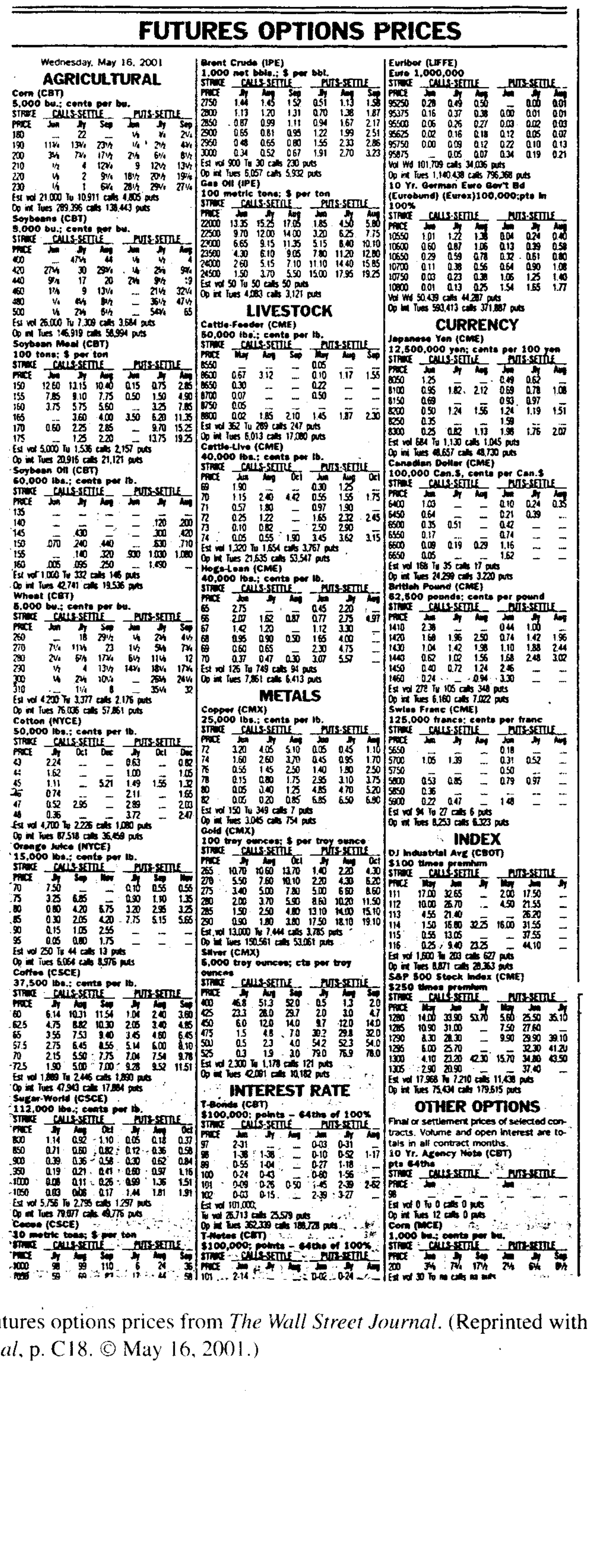
Futures options trade on many different exchanges, with underlying assets consisting of virtually every successful futures contract. The *Wall Street Journal* presents their prices in a table named "Futures Options Prices." Figure 14.8 illustrates how such futures options price data are presented.

Different conventions exist for interpreting futures options price data. Therefore, before you ever trade futures options, be sure to learn how to read the data. Let's look at some examples.

EXAMPLE 14.9 On November 4, 2001, a trader buys a call option on a December 2001 S&P 500 futures contract. The call costs \$8925, has a strike price of 1370, and expires on the Saturday following the third Friday of December (but the last trading day is actually the preceding Thursday). On that day, the December futures price is 1362.64. By buying this out-of-the-money call, the trader has the right to go long a December futures contract at a futures price of 1370. He will not want to exercise his call unless the futures price of the December contract rises above 1370. However, he can sell the futures call at any time.

Suppose that before expiration, the December S&P 500 futures price rises to 1425.00 and the futures call premium rises to \$14,000. The trader can sell his futures call for \$14,000, or he can exercise it. If he does the latter, he will then assume a long position in one December S&P 500 futures contract at a futures price of 1370. Since the futures settlement price for the contract on the day he exercises is 1425.00, his long position is marked to market, and he receives a cash inflow of $(250)(1425-1370)=\$13,750$. On subsequent days, his long futures position is marked to market until he offsets his futures position (or the last trading day is reached). Recall that S&P 500 futures are cash settled. Note that the trader receives more money by selling the futures call option than by exercising it.

If the trader does exercise his futures call, a writer is assigned the exercise. If the individual who is assigned is a naked writer, he must assume a short position in a March S&P 500 futures contract at a price of 1370. Because his position is immediately marked to market, he also has a cash outflow of \$13,750. If the individual who is assigned the futures call is a covered writer, he will deliver his futures contract at a futures price of 1370. He also will pay \$13,750. However, if he originally went long that contract at a futures price below 1425, he has realized some daily resettlement cash inflows on his long futures position.


FUTURES OPTIONS PRICES

Wednesday, May 16, 2001

AGRICULTURAL

Corn (CBT)

Soybeans (CBT)

Soybean Meal (CBT)

Soybean Oil (CBT)

Wheat (CBT)

Cotton (NYCE)

Orange Juice (NYCE)

Coffee (CSCE)

Sugar-World (CSCE)

Cocoa (CSCE)

Brent Crude (IPE)

Gas Oil (IPE)

LIVESTOCK

Cattle-Feeder (CME)

Cattle-Live (CME)

Hogs-Lean (CME)

METALS

Copper (CMX)

Gold (CMX)

Silver (CMX)

INTEREST RATE

T-Bonds (CBT)

T-Notes (CBT)

Euribor (LIFFE)

10 Yr. German Euro Gov't Bd (Eurobund) (Eurex)

CURRENCY

Japanese Yen (CME)

Canadian Dollar (CME)

British Pound (CME)

Swiss Franc (CME)

INDEX

DJ Industrial Avg (CBOT)

S&P 500 Stock Index (CME)

OTHER OPTIONS

Corn (MCE)

Figure 14.8 Some futures options prices from *The Wall Street Journal*. (Reprinted with permission of *The Wall Street Journal*, p. C18. © May 16, 2001.)

In Figure 14.8 the June 113 call on a Dow Jones Industrial Average futures contract has its premium listed as 21.40. However, note that at the heading says "$100 times premium." This means that the dollar cost of this futures call is ($100)(21.40) = $2140. This option no intrinsic value because the June futures settlement price on May 16, 2001 was 11,252 while the strike price of this call is 11,300. If the futures price subsequently rises above 11,300, then the call will be in the money. Note that the futures price for the June DJIA futures contract does not appear in Figure 14.8. It could be found in *The Wall Street Journal* for that day, however, in data like that shown in Figure 6.5. Note also that the underlying asset of the July DJIA futures options is a September DJIA futures contract. This illustrates an important fact that all users of futures options should investigate: many futures options have expiration dates in months that are not the same as the delivery months for the futures contracts that underlie them.

The June 101 put on a T-bond futures contract has a premium of 1-45. According to the header line of the table, this is 1 45/64% of $100,000. Thus, the dollar cost of this put is $1,703.125. This put consists of 1 18/32 (101 − 99 14/32) of intrinsic value and 9/64 (1 45/64 − 1 18/32) of time value, because the June T-bond futures settlement price was 99 − 14 (99 14/32: whereas T-bond option prices are quoted in 64ths, T-bond futures prices are quoted in 32nds) on that day. This put is in the money. The T-note futures option prices are read in the same way as T-bond futures options.

For the foreign exchange futures options on Japanese yen, Canadian dollars, British pounds, and Swiss francs, the premiums listed are presented as cents per unit of foreign exchange. That is, to compute the price of the futures option *in cents*, take the listed price and multiply it by the number of units of foreign exchange in a futures contract. Divide that figure by 100 to get the price in dollars. For example, the June 1450 British pound futures call has a listed premium of 0.40, or 0.40 cent per pound. In dollars, this is ($0.0040/£)(£62,500) = $250. This call is out of the money because the June British pound futures settlement price was $1.4294/£ on that day. Note that the June British Pound futures settlement price does not appear in Figure 14.8. It would have appeared, however, in *The Wall Street Journal* in data like that shown in Figure 6.5.

The Japanese yen futures options are a little different. Look at the header line of the table. It states that the prices are in "cents per 100 yen." This is actually the same convention that is used for yen futures price data, where the header line reads "$ per yen (.00)." The June 8200 put price is 1.24. The dollar price of this put is ($0.0124/100 yen)(¥12,500,000) = $1550.

The last trading day of futures options differs from contract to contract, and it can differ for a given contract, depending on whether the option expires in the futures' delivery month or in a non-delivery month. Always contact the exchange on which a futures option (or futures contract) trades for current contract specifications, since these can change at any time.

Futures options were illegal in the United States for long periods of time in the past. They were blamed for creating excessive volatility, and a subsequent collapse in grain prices, in July 1933. Because of that, the Commodity Exchange Act of 1936 made trading in options on "regulated commodities" illegal. However many commodities, such as metals, were not regulated. In the late 1960s and early 1970s options on spot gold and silver were sold to some investors. In addition, other firms bought commodity options in London (they were legal there) and sold them to American investors.

New scandals emerged as many investors found that they had grossly overpaid for these options. Some investment firms sold options they did not own, and when investors tried selling them or exercising them, the firms declared bankruptcy. The CFTC proceeded to ban trading in virtually all commodity options (options on individual stocks were legal, however) and futures options in June 1978. After considerable research, the CFTC rescinded its ban in 1982 and allowed futures options to trade on futures exchanges. The new futures options were regulated to minimize the likelihood of fraud and scandals.

14.16 Other Options

There are two broad classifications of "other options." The first consists of the set of options on various other underlying assets. For example, there is the huge OTC market for customized interest rate options (and options on debt instruments). When interest rate options are used to manage floating interest rate risk, they are called *caps* and *floors*, because they create a cap (a maximum) interest rate that will be paid in the future, or a floor (a minimum) on the interest rate that will be received. A cap is actually a call option on an interest rate, or a series of call options with future periodic expiration dates (such as quarterly or semiannually). A floor is actually a put option, or a series of puts, on an interest rate. Caps and floors are discussed in Chapter 19. Options on debt instrument are covered in Chapter 20.

Another interesting underlying asset is a swap. A **swaption** gives the owner the right to enter into a swap as either the fixed-rate payer or the fixed-rate receiver. Furthermore, since a swap is nothing but a portfolio of forward contracts, each having a different settlement date, a swaption is actually an option on this portfolio of forwards. In Chapter 20, we discuss swaptions. Related derivatives are the *callable swap*, which is a swap with the option giving the pay-fixed party the right but not the obligation to exit the contract, and the *puttable swap*, which is a swap that permits the receive-fixed party to terminate the contract. Be careful when using this terminology; be sure that both you and your counterparty understand the contract's terms in the same way.

The second broad classification of "other options" consists of *exotic* options, which are options that have unusual payoff mechanisms. In Chapter 20, we discuss several exotic options. Exotic options are generally classified into two groups: *path-dependent options*, which are options with pay offs that depend on the price path of the underlying asset before expiration, and *free-range options*, whose value is not path dependent.

Financial engineers are providing an endless stream of options that help manage risk. Interested readers should refer to Chapter 20, and also seek out other information on path-dependent and free-range exotics.[23]

14.17 Summary

This chapter is an introduction to options and options markets. An American call option gives the owner the right, but not the obligation, to buy the underlying asset at the strike price on or before the expiration day. The writer of a call has the obligation to sell the underlying asset, should he be assigned the exercise. An American put option gives the owner the right but not the obligation to sell the underlying asset at the strike price on or before the expiration date. The put writer is obligated to buy the put if he is assigned the exercise. European options can be exercised only on the expiration day.

The intrinsic value of an option is the greater of zero or its in-the-money amount. Time value equals the option's price minus its intrinsic value. At expiration, an option is worth only its intrinsic value. If option prices on the expiration day are not exactly $C_T = \max[0, S_T - K]$ and $P_T = \max[0, K - S_T]$, there will be arbitrage opportunities. Before expiration, options will almost always have some time value. The exceptions are deep in-the-money European options.

European call values C are higher for lower strike prices K, higher prices of the underlying asset S, longer times to expiration T (assuming no dividends), higher volatility of the underlying asset σ, higher interest rates r, and lower dividends prior to expiration. Put values P are higher if the strike price is higher, the price of the underlying asset is lower, the volatility of the underlying asset is higher, interest rates are lower, and dividends are higher. American puts cannot rise in value as time passes, all else equal, but it cannot be predicted how the value of a European put will change as its expiration day nears (all else equal).

Several features of options markets were described. Market makers at options exchanges quote bid prices, at which individuals can sell options, and asked prices, at which investors can buy options. The asked price always exceeds the bid price. Investors will typically either place market orders or limit orders to trade options. Market orders are likely filled at the current bid (if it is being sold by the individual) or the current asked (if the option is being purchased by the individual). Those who place market orders demand immediacy. In contrast, those who place limit orders are supplying liquidity, but they face the possibility that their orders to buy or sell options at a specific price will not be filled. The Options Clearing Corporation guarantees that the terms of all listed options will be fulfilled.

Finally, this chapter explained how to read prices in the financial press, and it briefly covered the topics of margin requirements, transaction costs, and taxes.

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Notes

[1]An index option is an option on a portfolio of stocks, such as the S&P 500, the S&P 100, or the Dow Jones Industrial Average.

[2]A futures option is an option on a futures contract.

[3]An OTC (over-the-counter) option is a custom-made option that does not trade on an organized exchange.

[4]Flex options were created by options exchanges in the United States to gain market share of the institutional option market. Institutions desired contracts that were custom made according to expiration date, strike price, and European vs American style. Flex options can be designed in these ways, and they trade on option exchanges.

[5]When a person who owns the underlying asset buys a put, he is purchasing insurance. If the value of the asset declines below the strike price, he will exercise his put and receive $$K$. He has insured against the possibility that the value of his underlying asset will fall below K. The price of the purchased put is the cost of insurance.

[6]This statement may not be true with respect to European calls on stocks that will pay dividends some time between the current day and the expiration day. It is always true for American calls, and for European calls on non-dividend-paying stocks.

[7]There are rules that define exactly who is entitled to distributions of cash and/or stock made by a firm. On a stock's ex-dividend day, the stock trades without the dividend. Thus an investor who buys a stock on its ex-dividend day is *not* entitled to receive that distribution. Prior to the ex-date, a stock is said to trade cum-dividend or with-dividend.

[8]We say "should" because in reality, stocks, on average, seem to fall by somewhat less than the dividend amount.

[9]This simple view of the world holds under specific assumptions. For example, it holds in a world of perfect capital markets and no taxes. Capital markets are said to be "perfect" when (a) there are no transactions costs; (b) securities are infinitely divisible, so that investors can buy fractions of securities; (c) information is costlessly available to everybody; and (d) investors are price takers, which means that their trades do not affect prices. See Chapter 3 of Fama and Miller (1972).

[10]In a stock split or stock dividend, a firm distributes x new shares for every old share an investor owns. For example, in a 2-for-1 split, an investor receives 2 new shares for every old share, thus doubling the number of shares owned. However, the price of the stock will fall by about 50%. Thus, an investor might own 100 old shares of a stock selling at $60/share before a split. On the ex-split date, he will own 200 new shares of a stock that will likely sell at $30/share. Stock dividends are usually smaller distributions than stock splits (e.g., a 5% stock dividend), and the two events are handled somewhat differently for accounting purposes.

[11]The exception to this statement is a European call on a stock that will trade ex-dividend prior to the expiration date. If the dividend amount is large, the call could be worth less than its current intrinsic value.

[12]There is an exception to this statement. In-the-money European puts can actually rise in value as time passes, all else equal. This phenomenon will be discussed in Chapter 15.

[13]Other variables that may play a role in option valuation include the skewness of the stock's distribution of returns, the probability that the stock will jump in value (a discontinuity in price), uncertainty about the future movements in interest rates, taxes, investors' risk aversion and/or beliefs (expectations) about future stock price movements, transactions costs, margin requirements for the options and for the underlying asset, and market microstructure (e.g., the level of competition among market makers, trading volume, and market-maker positions in the options).

[14]This diagram is a more accurate picture of American put pricing than of European put pricing. In-the-money European puts can sell for less than their intrinsic value and can also rise in value as time passes, a phenomenon that will be discussed in Chapter 16.

[15]Consider the range of possible stock prices for a typical $50/share stock tomorrow; you might believe the range is $49–$51. However, the dispersion of possible stock prices one week from today will likely be wider, perhaps $45–$50. One year from now, the stock might sell anywhere between $20 and $100. Thus, the longer the time horizon, the greater the dispersion of possible stock prices.

[16]The discussion in the text about the indeterminate relation between put values and time to expiration applies only to European puts. All else equal, American puts will never rise in value as time passes. If the second argument about the timing of the receipt of the sales proceeds becomes important, the holder of an American put would be well advised to exercise it early. Also, note that the first effect concerning the range of stock prices dominates when the put is out of the money. The second argument becomes increasingly important as the stock price falls, and the put becomes deeper in the money.

[17]In October 1990 the CBOE and AMEX initiated trading in options with times to expiration of up to 30 months. These options are called LEAPS (**long-term equity anticipation securities**).

[18]In other words, the customer of the clearing firm that wrote the option at the earliest date is assigned.

[19]There is some timing risk here. The option writer who is assigned an exercise may not learn of it for a day or two. During that time the stock price may change.

[20]If an investor writes a naked **index option**, the formulas for margin are slightly different. Instead of using 20% of the market value of the underlying stock, use 15% of the market value of the stock underlying the index.

[21]The CBOE publishes a useful booklet titled "Taxes and Investing: A Guide for the Individual Investor," which can be downloaded at the exchange's website (www.cboe.com/resources/tax.htm).

[22]A security's beta is its relative volatility compared to the market. If a stock tends to rise (fall) $x\%$ when the market rises (falls) $x\%$, then it has a beta of 1.0. A stock that tends to rise (fall) by 1 ½% when the market rises (falls) by 1% has a beta of 1.5. Beta is usually estimated by regressing the stock's returns on the market (the market model):

$$R_{it} = a_i + b_i R_{mt} + e_{it},$$

where

R_{it} = return on security i during period t
R_{mt} = return on the market during period t
a_i, b_i = regression coefficients; b_i is the estimate of security i's beta
e_{it} = a random error term

[23]See, for example, Konishi and Dattatreya (1996, Chapter 41), Wilmott (1998), or Hull (2000, Chapter 18).

PROBLEMS

14.1 There are six basic derivative positions:

1. Long forward
2. Short forward
3. Long call
4. Short (written) call
5. Long put
6. Short (written) put

Figure P14.1 shows the six profit diagrams for the six positions. In each diagram, the profit is on the y-axis (the vertical axis) and the expiration day price of the underlying asset is on the x-axis. Match each diagram (a–f) with the appropriate position listed above.

14.2 Which is worth more: a deep in-the-money call or a deep out-of-the-money call?

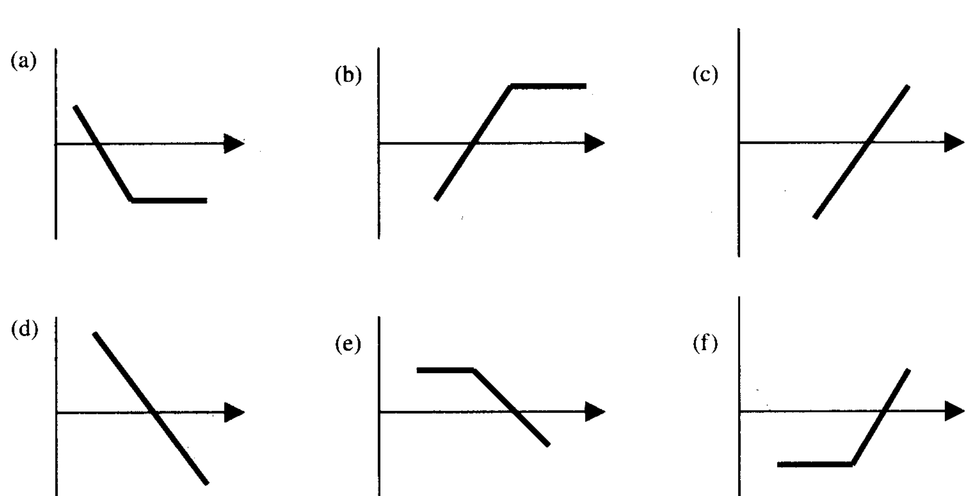

Figure P14.1.

14.3 Explain why ex-dividend days will make calls less valuable and puts more valuable, all else equal.

14.4 Suppose you owned a call, and after the close of trading, the company announces that its next dividend will be considerably greater than you, or any one else, expected. The next ex-dividend date falls prior to your call's expiration date. Will your call open higher or lower the next day?

14.5 Refer to the America Online (AmOnline) option price data in Figure 14.5.

a. Are the calls with a strike price of 55 in the money or out of the money? Why?

b. Are the puts with a strike price of 55 in the money or out of the money? Why?

c. What is the intrinsic value of the July 50 calls? What is the time value of the July 55 calls?

d. What is the intrinsic value of the July 50 puts? What is the time value of the July 55 puts?

e. Suppose you bought one July 55 call at the price shown, and at the close of trading of the expiration day, America Online closed at $57/share? What would be your profit or loss? What would be your holding period rate of return? What would be your annualized rate of return?

f. Suppose you wrote one July 50 naked put at the price shown, and at the close of trading of the expiration day, America Online closed at $43. What would be your profit or loss? Under what conditions would the owner of a July 50 put exercise the put at expiration?

14.6 How is it possible for the value of a call to rise from one day to the next if the stock price remains unchanged on those two days?

14.7 An option is quoted at "2.70 bid–3 asked." What are the potential benefits and costs of placing a limit order to buy the option at 2.70 versus placing a market order to buy?

14.8 Suppose that at expiration, a stock is quoted at "44.10 bid–44.20 asked." A put option with a strike price of 45 is quoted at "0.90 bid–1.00 asked." Demonstrate whether an arbitrage opportunity does or does not exist. Suppose that the put is quoted at "0.60 bid–0.70 asked." Demonstrate whether an arbitrage opportunity does or does not exist.

14.9 What are the six fundamental determinants of option values? Predict how call values change (increase or decrease) as each of the six parameters decreases, all else equal. Predict how put values change as each of the six parameters decreases all else equal.

14.10 Refer to Figure 14.5. What would be the required margin for writing a naked Cisco July 25 put? Why would an individual want to write this naked put? In general, compare the benefits, risks, and costs to these two strategies: writing a naked put versus buying a call.

14.11 Which is worth more: an American option or a European option? Why?

14.12 In Figure 14.5, refer to the AmOnline and ApdldMat calls with a strike price of 50 that expire in June. Discuss the possible reasons that the ApdldMat calls sell for more than the AmOnline calls.

14.13 Refer to the Broadcom options in Figure 14.5. Use the information to infer whether Broadcom common stock has been generally rising or falling over the past several months.

14.14 Explain why an increase in volatility increases the range of both good and bad outcomes for a stock, yet it unambiguously increases the value of both put and call options.

14.15 What makes index options different from ordinary options on individual equities?

14.16 Refer to Figure 14.8 to answer the following questions.

a. What is the call premium for the June 8100 call on a Japanese yen futures contract? What is the underlying asset of this futures option?

b. The yen futures price for the contract deliverable into the June futures options was \$0.008126/¥. In dollars, what are the intrinsic value and the time value of the June 8100 futures call? What are the intrinsic value and the time value of the June 8200 futures call? What are the intrinsic value and the time value of the July 8050 and July 8300 puts on yen futures?

c. If a yen futures option premium is \$187.50, what price would you see in the presentation of futures options price data, as in Figure 14.8?

14.17 Which of the following correctly describes the situation of the writer of a call option?

a. has the right to sell the underlying asset at the strike price.

b. has the right to buy the underlying asset at the strike price.

c. has the obligation to sell the underlying asset at the strike price if the call owner exercises.

d. has the obligation to buy the underlying asset at the strike price if the call owner exercises.

e. has both the right and the obligation to sell the underlying asset at the strike price.

f. has both the right and the obligation to buy the underlying asset at the strike price.

14.18 The stock price is 37. A put with a strike price of 35 has a premium of 3. Which of the following is true?

a. The intrinsic value is 2 and the time value is 1.

b. The intrinsic value is 3 and the time value is 1.

c. The intrinsic value is 5 and the time value is 1.

d. The intrinsic value is 0 and the time value is 3.

e. The intrinsic value is 2 and the time value is 3.

f. The intrinsic value is 5 and the time value is 3.

14.19 Which of the following offers an investor the chance to make an unlimited profit if the underlying asset rises in price, but with limited losses if the underlying asset declines in price (i.e., the position requires only a small initial investment, which is the most that an investor can lose)?

a. Going long a forward contract.

b. Buying a call.

c. Buying a put.

d. Writing a call.

e. Writing a put.

CHAPTER 15

Option Strategies and Profit Diagrams

A **profit diagram** depicts the profits and losses from an option investment strategy as a function of the price of the underlying asset at expiration. Profits and losses are on the y axis, and a range of expiration date (time T) prices of the underlying asset are on the x axis. These diagrams are sometimes called payoff patterns or payoff profiles.

The payoff pattern to a long position in an asset is linear. For example, if you own one share of stock and its prices rises by a dollar, your profit is a dollar. In contrast, options offer nonlinear payoff patterns. If you own a call option on one share of stock then, at expiration, your profit rises by one dollar for every one-dollar increase in the stock price *only if the option is in the money.* If the option is out of the money, your profit is fixed (constant) for any expiration day price of the underlying asset; that is, you have lost the initial premium you paid for the option, and this is independent of how much the option finishes out of the money. Thus, a long call offers two piecewise linear payoff segments. Even more unusual payoff patterns are created when several different options are used, perhaps combined with the underlying asset. Some of these strategies have colorful names: spreads, straddles, strips, straps, butterflies, and condors, for example. This chapter presents many different option strategies and their profit diagrams.

In most of this chapter, we assume there are only two relevant dates for the purchase and/or sale of the assets involved: the initial date (time 0) and the expiration date (time T). Nothing, however, prevents an investor from closing out positions early. We also ignore the possibility that the options will be exercised early. Dividends, commissions, and margin requirements are not explicitly dealt with, but can easily be incorporated into one's preparation of profit diagrams. Remember, too, that because of nonsynchronous trading and bid–asked spreads, prices in the financial press may not reflect prices at which an investor can buy or sell the options. Finally, our discussion focuses on nominal dollar cash flows. This means that the timing of cash flows is ignored—we will simply add dollar cash flows that occur at two dates. Although this is a common practice in preparing profit diagrams, in concept, it is sacrilegious. We all know that a dollar inflow at time 0 is more valuable than a dollar received later.

Profit diagrams can provide additional valuable information if they are prepared on a rate-of-return basis, and also if they are prepared showing profits and losses that result when positions are closed at dates earlier than expiration. These two types of diagram will be illustrated at the end of the chapter.

Table 15.1 contains the notation that will be used in this chapter.

15.1 Profit Diagrams for Long Stock and Short Stock

The easiest way to illustrate a profit diagram is to begin with the two basic positions in the underlying asset. An investor can buy the asset today for S_0 and sell it at a future date at an unknown

TABLE 15.1 Notation

Variables	Subscripts
S = stock price	0 = today (strategy initiation date)
K = strike price	T = expiration date
C = call premium	H = high strike price
P = put premium	L = low strike price
T = time to expiration	

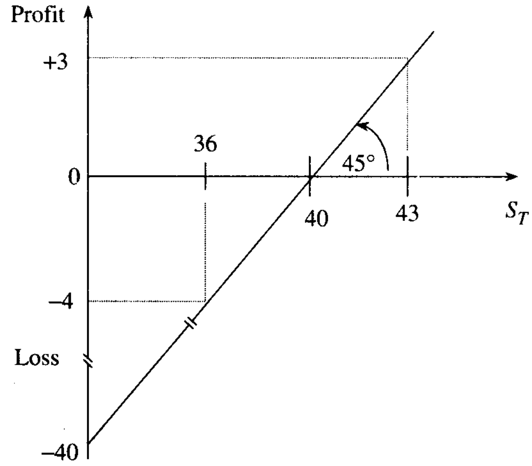

Figure 15.1 Long stock.

price $\tilde{S}_T$. Once purchased, the investor is said to be "long" the asset. In this chapter, we will assume that the underlying asset is 100 shares of common stock, but it could as easily be 100 ounces of gold, $1 million of Treasury bills, 6.25 million Japanese yen, a futures contract, or an office building.

The profit diagram for a long position in stock is shown in Figure 15.1: the stock is purchased at S_0=$40/share. If, at a later date, the stock is sold at a price of S_T=$43/share, the investor will earn a profit of $3/share. Similarly, if the stock is later sold for $36/share, a loss of $4/share will be realized. The maximum loss (the y-axis intercept) is $40/share, which occurs if S_T falls to $0. Note that the profit diagram is a straight line that passes through the $40 point on the x axis, creating a 45° angle; thus, if the stock is later sold for $40 (the purchase price), a $0 profit results. It is clear from the diagram that a stock purchase is a bullish strategy because profits will be realized from price appreciation.

Investors can also sell stock short. The profit diagram for a short sale of stock is presented in Figure 15.2. In this type of trade, an investor sells the stock today and later repurchases it. However, the investor does not initially own the stock that is sold. Rather, the stock is borrowed from someone else. When the stock that was sold is later repurchased, it is returned to the original owner. Selling short is a bearish strategy because profits accrue when the underlying asset price declines.[1]

Observe Figure 15.2 and assume that the stock is sold short at S_0=$50/share. If the stock is later purchased at the same price, there is no profit or loss. If the stock declines in price, profits are made. The profit is $1 per share for every dollar decline in price. If the stock price falls to zero, the maximum profit of $+\$S_0$ will be earned; if $S_0=50$, then the maximum profit is $50. If the stock price declines to $46, the short seller's profit is $4/share. For smaller stock price declines, the profit is S_0-S_T, where S_T is the purchase price at time T. If the stock price rises, however, the investor loses. The potential loss is theoretically infinite. While infinite losses have never been observed, short selling can lead to losses many times over the original stock price.

Summarizing, the profit diagrams for stock-only positions are seen to be straight lines that pass through the original transaction price with zero profit. The line for the long stock position has a slope of +1, while that for the short stock position has a slope of −1. Option positions create many other payoff patterns.

15.2 Long Calls

Figure 15.3 shows the profit diagram for the purchaser of a call option. Here, the strike price is $50/share and the call premium is $3/share. If, at expiration, the stock price is $50 or lower, the option expires worthless. The call buyer will lose the premium paid of $3/share ($300 for a call on 100 shares). If the stock price closes on the expiration date at 53, the call buyer breaks even. He initially paid $3 for the call, and at expiration, the call is worth $3 [Recall that at expiration, a call is worth $C_T=\max(0, S_T-K)$.] At any stock price above $53, the call buyer will realize a profit. For every $1 increase in the stock price, the call is worth $1 more. Thus, the diagonal portion of the line in Figure 15.3 has a slope of +1.

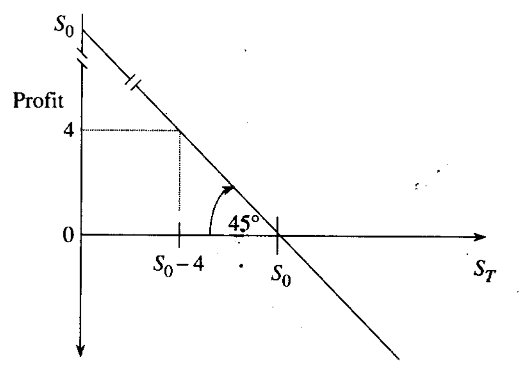

Figure 15.2 Short stock.

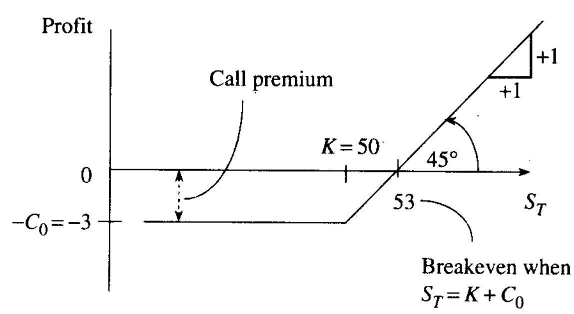

Figure 15.3 Long call.

It is clear from Figure 15.3 that buying a call is a bullish strategy with respect to the underlying asset. If the stock price remains unchanged (or declines), the buyer of an at-the-money call or an out-of-the-money call will lose 100% of his initial investment. If the call buyer is to profit, the expiration day price of the underlying asset must rise above the strike price to cover the initial price paid for the call. The buyer of an out-of-the-money call will profit only if the stock price rises by more than the call premium *plus* the amount by which the call is originally out of the money. For example, if $K=50$, $S_0=46$, and $C_0=2$, the stock price must rise by \$6 at time T for the call buyer to break even. Brokerage commissions on the purchase and the sale of the option add to the amount the stock must rise to merely break even.

15.3 Writing a Naked Call

Figure 15.4 depicts the profit diagram for the writer of a naked call. The writer of a naked call does not own the underlying security.

Figure 15.4 is a general example of a profit diagram because algebraic symbols are used instead of specific numerical values. The strike price is K. The original call premium received by the seller is C_0. If the stock price at expiration S_T is at or below K, the option expires worthless and the call writer keeps the premium. Thus, there is a horizontal line with a profit of $+C_0$ for all $S_T \leq K$ in Figure 15.4. If $S_T > K$, the call will have value at expiration, and the call writer will have to buy back the call for the amount $C_T = S_T - K$. Thus, for every dollar the stock price rises above K, the call writer loses a dollar, and the diagonal portion of the line has a slope of -1. The break-even point (\$0 profit) occurs when $S_T = K + C_0$. In this case $C_0 = C_T$.

Writing a naked call is a bearish strategy.[2] If the stock remains unchanged or declines, the writer of an at-the-money or out-of-the-money naked call will keep the premium and the interest earned on the premium over the time she was short the call. However, the writer should also be aware of potentially substantial margin requirements on naked option positions. The call writer must also be aware that there are substantial risks if one is wrong about the future price movement of the stock. Observe in Figure 15.4 that losses are potentially unlimited as the stock price rises.

Call writers can write at-the-money calls, out-of-the-money calls, or in-the-money calls. When writing out-of-the-money calls, one will still keep the premium if the stock rises in price somewhat or remains unchanged. The risk of being assigned an exercise is less, but the out-of-the-money call premium will be smaller than the premium of an at-the-money call, so the potential

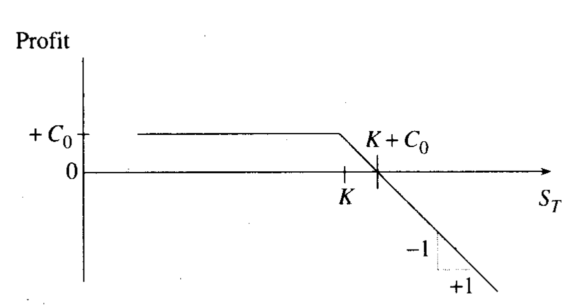

Figure 15.4 Writing a naked call.

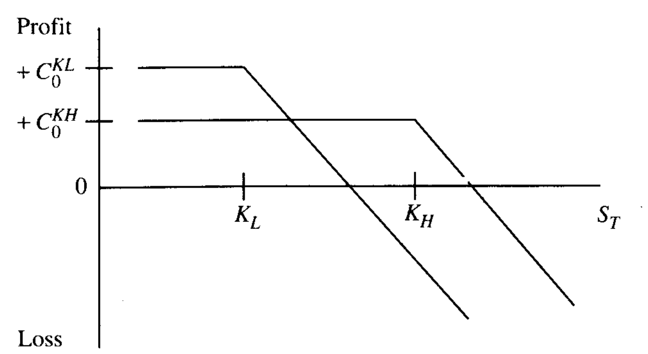

Figure 15.5 The trade-offs associated with writing calls with different strike prices. If a call with a high strike K_H is written (perhaps an out-of-the-money call), a smaller premium, C_0^{KH}, is received, but there is a larger range of expiration day prices of the underlying asset for which the call writer gets to keep that premium as profit. In contrast, writing a call with a lower strike price K_L provides a larger premium of C_0^{KL}, for the call writer; however there is a greater probability that this call will finish in the money and therefore create a loss for the call writer.

profit is smaller. This concept is illustrated in Figure 15.5. Here, a call writer may write an at-the-money call for a premium of C_0^{KL} at a strike price of $S_0 = K_L$. Alternatively, an out-of-the-money call may be written with a higher strike price of K_H. A smaller premium (C_0^{KH}) is received, but there is a greater probability that the call writer will keep that premium.

15.4 Long Puts

Figure 15.6 illustrates the long put profit diagram. If $S_T > K$, the put closes out of the money at expiration, and the put buyer will lose the price initially paid for the put, P_0. If the stock price at expiration is less than the exercise price, the put will have value at time T. Recall that $P_T = 0$ if $S_T \geq K$, and $P_T = K - S_T$ if $S_T < K$. If S_T is \$1 below K, the put is worth \$1 at expiration. The put buyer will break even (zero profit) if S_T equals $K - P_0$. The maximum profit is earned if $S_T = 0$. The put will then be worth K at expiration, and the profit equals $K - P_0$.

Buying a put is a bearish strategy. If an at-the-money put is bought and the stock price remains unchanged (or rises), the investor loses 100% of the initial investment. The benefit to this strategy is that the put buyer can lose only the initial premium paid. Contrast this to the short seller of the stock, who is exposed to theoretically unlimited losses (see Figure 15.2). In addition, a considerable amount of margin funds may be required in selling stock short, while a put purchase requires an outlay consisting of only the cost of the put (plus commissions).

15.5 Writing a Naked Put

Suppose that an instant before the close of the market on May 10, 2001, an investor writes a naked put option on Applied Materials (ApdldMat in Figure 14.5). More specifically, the put is the June

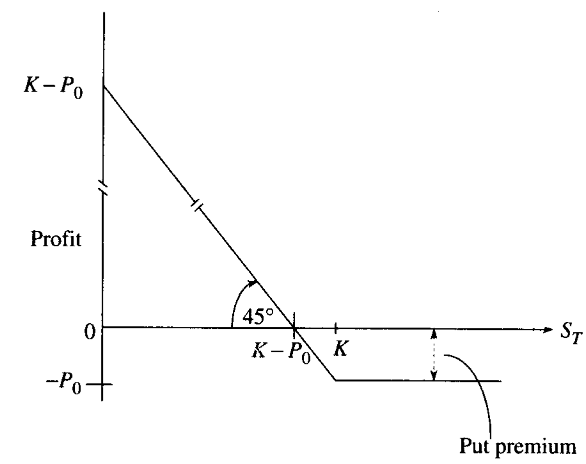

Figure 15.6 Long put.

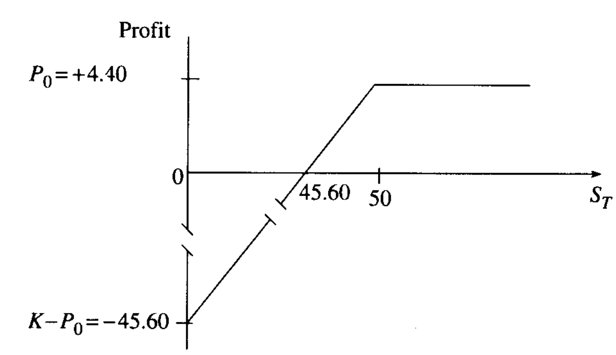

Figure 15.7 Writing a naked put.

50 put, and the premium received is 4.40 ($440 for a put on 100 shares). The profit diagram for the written put position shown in Figure 15.7 uses some numbers from the *Wall Street Journal* reprint shown earlier (Figure 14.5). If AMAT (the ticker symbol for Applied Materials) closes at a price of $50 or higher on the third Friday in June, the put will expire worthless. In this case, the put writer keeps the $4.40 per share. Therefore, Figure 15.7 shows a $4.40 profit for all $S_T > 50$. If, however, AMAT closes at $45.60 at expiration, the put writer breaks even ($S_T = K - P_0$). At lower prices, the put writer loses $1 for every dollar that AMAT's stock price falls. If AMAT's stock were to fall to zero in June (the probability of this happening is extremely small, to say the least), the maximum loss of $K - P_0 = \$45.60$ would be realized. This is shown at the y-axis intercept.

Writing a naked put is a bullish position with respect to the underlying stock, and the writer must be prepared to accept ownership of the shares, should the put owner decide to exercise.[3] In the foregoing example, the risk of early exercise on May 10 (the day of the prices shown in Figure 14.5) is zero, because AMAT's stock price is $51.01. Because $S_0 > K$, the put is out of the money. But, should AMAT fall below $50/share, the possibility of early exercise could increase because the put moves in the money. If AMAT is below 50 at expiration, the put owner will most certainly exercise the option.[4]

The name **cash-secured written put** is given to the strategy in which the investor writes a naked put and invests the strike price amount in riskless securities. Usually, an out-of-the-money put is used. Thus, if AMAT is at $51.01/share, writing a naked put with a strike price of 50 is a substitute for a limit order to buy AMAT at 50. There are, however, some differences:

a. Once AMAT has fallen to a price of 50, the limit buy order will be executed at that price or lower. In contrast, the put writer may observe AMAT's price temporarily fall below 50; but if the put owner elects not to exercise, the put writer will not get to purchase the stock. If the stock then proceeds to rise, the put writer may never be assigned an early exercise and will regret not having placed a limit order to purchase the stock.

b. Suppose that an imbalance of too many sell orders results in the temporary suspension in trading in AMAT's stock. Perhaps AMAT's earnings were disappointing, or the firm unexpectedly lost a lawsuit. AMAT's price could conceivably reopen for trading at, say, $30/share, when the last trade on the previous day was at $51.01/share. Had a limit order to buy at $50 been placed, the investor would have had the pleasure of knowing that his purchase price would be $30. The naked put writer, however, would have been forced to pay $50/share.

Keep in mind that the put writer is paid $4.40 (see Figure 14.5) for the June 50 put to assume these risks over the upcoming five weeks (until the expiration day).

Many other payoff patterns can be created with various long and/or short positions in calls, puts, and/or the underlying asset. Several of these will now be illustrated, and several others are outlined later in this chapter. Note, however, that one popular strategy called the horizontal spread, also known as a time spread, cannot be covered until we have learned how to value options.[5]

15.6 Covered Call Writing

A covered call is an example of a "hedge position," where an option provides some compensation for the risk that the price of the underlying asset will decline. When writing a covered call, the investor buys (or already owns) 100 shares of stock and writes a call. The purpose of the strategy is to generate income from the proceeds of the written call. However, we will see that the covered call writer is effectively selling some of the upside potential of the stock in exchange for the premium.

The easiest way to prepare a profit diagram for a covered call, or for any complicated strategy, is to first prepare a profit table. This is done in two steps:

a. Calculate the initial inflow or outflow.

b. Calculate the expiration values of the assets involved, given several potential expiration day stock prices around the relevant strike prices of all options that are used in the strategy.

For example, refer again to Figure 14.5, and suppose that an investor buys 100 shares of Cisco for $18.83/share and writes 1 July 20 call option for $1.90 per call on one share. Initially:

Buy 100 shares of Cisco	−18.83 per share	Cash out: −$1883.00
Write 1 July 20 call	+1.90 per share	Cash in: $190.00
Initial cash flow at time 0, CF_0	−16.93 per share	Net cash out: −$1693.00

Next, consider a range of possible prices of the underlying asset on the expiration day, T. Then, compute the cash flows by reversing all open positions at time T:

Price of Cisco, S_T	Sell Cisco	Buy (offset) July 20 Call for $C_T = \max[0, S_T - K]$	Time T Cash Flows, CF_T	Profit/Loss $CF_0 + CF_T$
14	+14	0	14	-2.93
15	+15	0	15	-1.93
16	+16	0	16	-0.93
17	+17	0	17	+0.07
18	+18	0	18	+1.07
19	+19	0	19	+2.07
20	+20	0	20	+3.07
21	+21	-1	20	+3.07
22	+22	-2	20	+3.07
23	+23	-3	20	+3.07

At any expiration day price at or below 20, the call expires worthless. The investor sells the shares for whatever closing price occurs for Cisco on the expiration date, S_T. At any price above 20, the call finishes in the money, so that the covered call writer will have to repurchase the option for $S_T - K$. However, the covered call writer will also be able to sell Cisco at the higher price.[6] The result is that for every dollar increase in Cisco's stock, the investor profits by \$1 because of the long position in the stock, but loses \$1 because the call was written. In the last column in the foregoing tabulation, we add the time T cash flow, CF_T, to the initial cash outflow of $CF_0 = -16.93$ per share. This determines the profit or loss from the strategy.

Actually, because all these profit diagrams produce straight lines that may be "kinked" at the strike prices, we need only consider time T prices around the strike price. In this example, we only had to prepare a table for $19 \leq S_T \leq 21$. For prices below 19 and above 21, we can extrapolate when preparing the profit diagram. An important point to remember is that in expiration date profit diagrams, kinks occur only at exercise prices. In this example, a kink occurs only at $K = 20$.

The profit diagram for a covered call on Cisco, using the July 20 call, is shown in Figure 15.8. The zero-profit (breakeven) point occurs when $S_T = \$16.93$. At that closing price at expiration, the loss on the stock $(16.93 - 18.83)$ equals the proceeds from the written call (1.90).

An alternative method of preparing profit diagrams is to consider only the basic positions and then vertically add them.[7] This approach is shown in Figure 15.9, where we see that at prices of 20

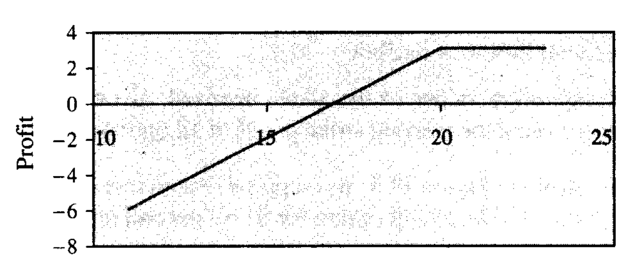

Figure 15.8 Writing a covered call. In this covered call strategy, $S_0 = \$18.83$, $C_0 = \$1.90$, and $K = \$20$. The maximum profit of \$3.07 is realized when $S_T \geq \$20$. The break-even point (zero profit) is at $S_T = \$16.93$.

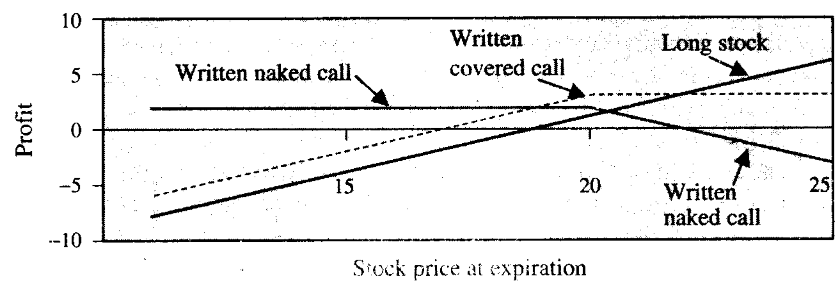

Figure 15.9 Alternative approach to preparing a profit diagram for a covered call.

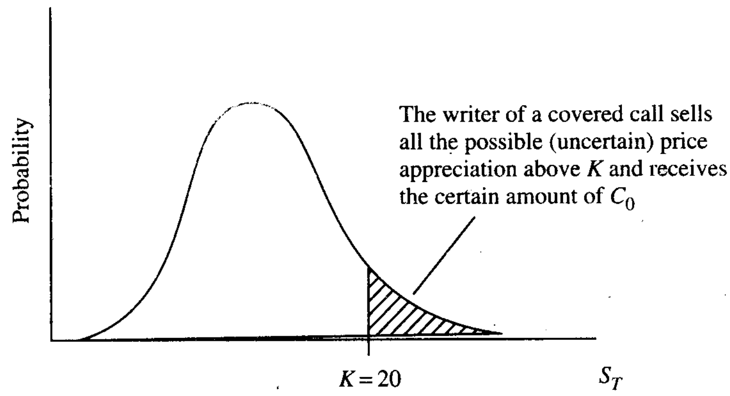

Figure 15.10 Writing a covered call.

and above, there are two 45° lines, one with a +1 slope (long stock) and one with a −1 slope (short call). Adding these creates the horizontal-line portion of the covered call shown with a profit of +3.07 when $S_T \geq 20$ in Figure 15.9. As we move below 20, the investor loses on the long stock position but gets to keep the call premium. Thus, we add a constant amount (slope = 0), the call premium, to the long stock position, and the result is the diagonal line in Figure 15.9, with a slope of +1, when $S_T < 20$.

In effect, the writer of covered call sells off all (risky) price appreciation potential above K in return for a riskless cash inflow, the call premium. This is illustrated in Figure 15.10.

Often, a writer of a covered call believes the stock is attractively priced and offers some upside potential for price appreciation. Thus, the chosen strike price will usually lie above the current stock price; that is, out-of-the-money calls are generally written. If the price rises, the writer of a covered call will capture some of that price increase (up to the exercise price), as well as keep the call premium. If the investor is wrong, and the stock declines, the losses are tempered somewhat by the call premium that is kept.

However, keep in mind that if the stock price rises sharply, the writer of a covered call will regret having written the call. Any price appreciation above K has been sold. Also, if the stock price falls sharply, the writer will regret having bought the stock. If, for example, Cisco were to plummet to $10/share between the date the call was written and the third Friday in July, the investor with a covered call position still loses a substantial sum.

Covered call writing on stocks that pay high dividends can be an attractive cash management tool for corporations. Usually, corporations pay taxes on only 30% of the dividends they receive from their ownership in the shares of other firms. Corporations can buy the shares before an ex-dividend date and write the calls. On the expiration date, the positions are reversed.[8]

Comparison reveals that Figures 15.7 and 15.8 are almost identical. Actually, a written naked put is the same as a covered call position, except for a scale factor. In Chapter 16, we will see that put–call parity on European options with no dividends explains why the positions are equivalent and that the scale factor is the present value of K.[9]

Finally, there is no need to write only one call against the 100 shares of stock. **Variable-ratio hedges** allow the selling of any number of options against the 100 shares the investor owns, and each of these strategies leads to a different profit diagram. Indeed, an investor can write an appropriate number of calls per 100 shares of stock to maintain a riskless hedge (the position yields the risk-free rate of interest) for small movements in the underlying stock. This idea will be covered in Chapters 17 and 18, and it has great significance for the derivation of the Black–Scholes option pricing model, which is discussed in Chapter 18. Other variable-ratio hedges might involve the purchase of puts as protection against price declines in the underlying long stock position, or "reverse" hedges where stock is sold short and either calls are bought or puts written. Examples of these strategies are posed as problems at the end of this chapter.

15.7 Vertical Spreads

Vertical spreads are also called money spreads or price spreads. The use of the term "vertical" arose because of the format in which option price data formerly were presented in the financial press: different strikes were listed vertically. In a vertical spread, an option with one strike price is bought, and another option with a different strike price (but the same time to expiration)[10] is sold. One can create a bullish vertical spread with puts or with calls. Similarly, a bearish vertical spread can be created with puts or with calls.

15.7.1 Bullish Vertical Spread with Calls

You are surfing www.cboe.com and see a Jan 80 Home Depot call selling for $3.75. Suppose you felt there was a high probability that Home Depot shares would sell for at least $80 by the third Friday in January. If you purchased the Jan 80 call and the stock closed right at $80 on the expiration date, you would lose your $3.75 call premium. You also see a Jan 75 call selling for 5, so if you bought this call, and if $S_T = 80$, you would still not profit. (the initial cost of the call is $500, and the call would be worth $S_T - K = \$500$ at expiration, so the profit is zero).

A bullish vertical spread lowers your initial outlay to only $125, and at any $S_T \geq 80$, your proceeds at expiration would be $500. This is a 300% return on your initial investment. An investor buys this vertical spread by buying a call with the lower exercise price, $K_L = 75$, and selling a call with a higher exercise price, $K_H = 80$. In our example:

Buy 1 Jan 75 Home Depot call	-5
Sell 1 Jan 80 Home Depot call	$+3.75$
Initial cash flow at time 0	$-1.25 = CF_0$

The expiration day profit table is then as follows:

S_T	Time T: Sell Jan 75 Call	Time T: Buy Jan 80 Call	Cash Flow, CF_T	Profit/Loss CF_0+CF_T
74	0	0	0	−1.25
75	0	0	0	−1.25
76	+1	0	+1	−1.25
77	+2	0	+2	+0.75
78	+3	0	+3	+1.75
79	+4	0	+4	+2.75
80	+5	0	+5	+3.75
81	+6	−1	+5	+3.75
82	+7	−2	+5	+3.75

At all prices ≤75, both calls expire worthless. At all prices ≥80, the cash inflow at expiration, CF_T, is $500. Figure 15.11 is the profit/loss diagram for this example of a bullish vertical spread using calls.

You might try drawing profit/loss diagrams for a bullish Home Depot call spread using the January 70 and January 80 calls. The premium for the Jan 70 call is 7.40. Then, compare the resulting diagram and Figure 15.11 to another diagram generated by buying the January 70 call and selling the January 75 call.

15.7.2 Bullish Vertical Spread with Puts

In a bullish vertical spread with puts, the investor writes a put option having a high exercise price K_H and buys a put with a low exercise price K_L. Consider the Jan 75 and 80 puts on Home Depot. Initially, suppose:

Write 1 Jan 80 Home Depot put	+13
Buy 1 Jan 75 Home Depot put	−10.8
Initial cash inflow	$+2.2=CF_0$

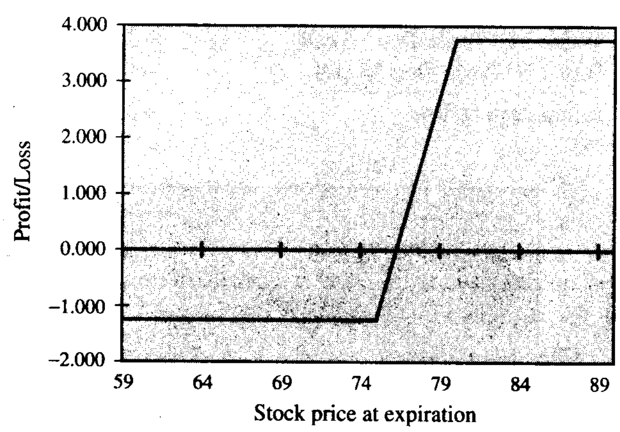

Figure 15.11 Bullish vertical spread with calls: $K_H=80$, $C_H=3\ 3/4$, $K_L=75$, $C_L=5$. The low strike call is bought, and the high strike call is sold.

Then, at expiration, we have:

	Time T			
S_T	Buy Jan 80 Put	Sell Jan 75 Put	CF_T	Profit/Loss, CF_0+CF_T
74	−6	+1	−5	−2.8
75	−5	0	−5	−2.8
76	−4	0	−4	−1.8
77	−3	0	−3	−0.8
78	−2	0	−2	+0.2
79	−1	0	−1	+1.2
80	0	0	0	+2.2
81	0	0	0	+2.2

In this table, CF_T is the expiration day cash flow generated by the repurchase of the Jan 80 put and the resale of the Jan 75 put; these offset the original trades. The total profit, shown in the last column, is depicted in Figure 15.12.

Note that this bullish spread with puts has an added element of risk if the written put is in the money. Thus there is risk that the written in-the-money put can be exercised early by its owner. However, should the investor's expectations turn out to be correct, both puts will expire worthless, there will be no exercise, and transactions costs will be saved.

In contrast, for the bullish spread with calls, the calls currently have no risk of early exercise; but should Home Depot's stock price rise above 80, the investor will have to pay commissions on the resale and repurchase of the two calls. If Home Depot were to trade ex-dividend between the day the strategy is executed and the expiration day of the options, and if Home Depot were selling for more than $80/share just prior to that day, the investor would bear the risk that the written call would be exercised.[11] Thus, there are benefits and risks to each bullish strategy.

15.7.3 Bearish Vertical Spread with Calls

Bearish vertical spreads with calls are created by the sale of a call with a low exercise price K_L and the purchase of a call with a high exercise price K_H. For example, suppose E*Trade.com option prices allow the following:

Write 1 E*Trade Dec 30 call	+3.4
Buy 1 E*Trade Dec 35 call	−1.3
Initial cash inflow	$+2.1=CF_0$

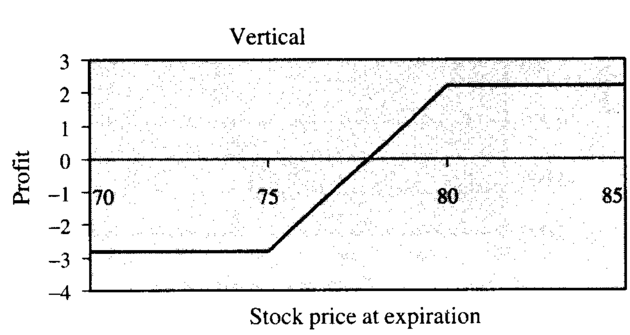

Figure 15.12 Bullish vertical spread with puts: buy the $K=75$ put and sell the $K=80$ put.

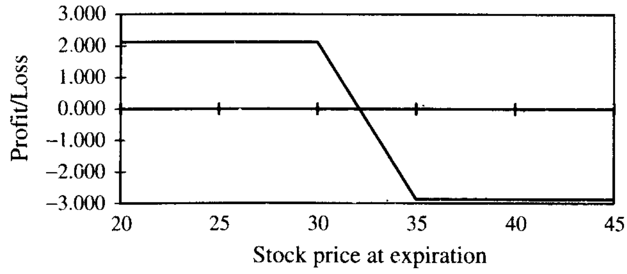

Figure 15.13 Bearish vertical spread with calls: the $K_L = 30$ call is sold and the $K_H = 35$ call is bought.

Continuing with the profit table, at time T, we find the following:

S_T	Buy Dec 30 Call	Sell Dec 35 Call	Time T Cash Flow CF_T	Profit/Loss $CF_0 + CF_T$
29	0	0	0	+2.1
30	0	0	0	+2.1
31	−1	0	−1	+1.1
32	−2	0	−2	+0.1
33	−3	0	−3	−0.9
34	−4	0	−4	−1.9
35	−5	0	−5	−2.9
36	−6	+1	−5	−2.9
37	−7	+2	−5	−2.9

Figure 15.13 shows the profit diagram for this bearish vertical spread using the E*Trade calls.

15.7.4 Bearish Vertical Spread with Puts

To create a bearish vertical spread with puts, a put with a high exercise price, K_H, is bought and a put with a low strike price, K_L, is written. This results in an initial cash outflow of $P(K_H) - P(K_L)$, because the put with the high strike price costs more than the proceeds generated from writing a put with a low strike price. If both puts finish in the money, they will be worth $K_H - S_T$ and $K_L - S_T$, respectively, and the cash inflow at time T equals $K_H - K_L$. The total profit $(CF_0 + CF_T)$ will be $(K_H - K_L) - [P(K_H) - P(K_L)]$. If both puts finish out of the money, there is no cash flow at time T, and the total loss equals the initial cash outflow, $P(K_H) - P(K_L)$. The final profit diagram looks like Figure 15.14.

The reader is urged to work out an example of a bearish vertical spread with puts using price data from Figure 14.5, or recent option prices from the *Wall Street Journal* or Internet sources (such as www.cboe.com).

15.7.5 Other Thoughts on Vertical Spreads

Some simple logic might make the spread terminology easier to remember.

- Whenever you experience an initial cash outlay with puts (e.g., you buy a put spread by purchasing an expensive put and selling a cheap put), you must be bearish. Just recall that if you bought a put (a cash outflow), you must be bearish.

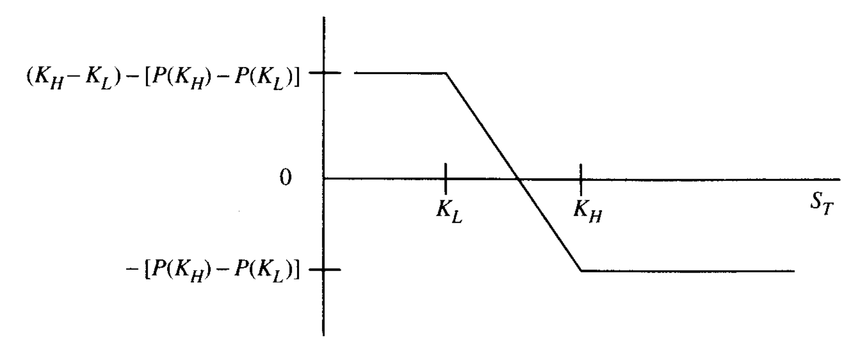

Figure 15.14 Bearish vertical spread with puts.

- If you were to buy a call spread, you experience a cash outlay at time 0. In this case (just as when you buy any call), you must be bullish. To buy a bullish call spread, you buy an expensive (low K) call and sell a cheaper (high K) call.
- The seller of a call is bearish, and an initial cash inflow is received. When you sell a call spread you are bearish. Selling a vertical spread with calls entails a cash inflow at time 0. This means you must sell an expensive call and buy a cheaper call to set up a bear spread with calls and generate a cash inflow.
- Finally, consider the seller of a put spread. There is a cash inflow at time 0. A naked put seller also has a cash inflow at time 0 and is bullish. Thus, the seller of a put spread must be bullish. To sell a put spread, realize a cash inflow at time 0 by selling an expensive put and buying a cheaper put.

There are several situations that would lead an investor to use vertical spreads. First, a vertical spread should be placed if an investor believes that the stock price will move only to the strike price that generates the maximum profit. For example, either of the two bullish spreads involving Home Depot should be used if an investor believes the stock will rise only to $80 or a little higher. An investor who is very bullish should just buy the calls.

A bullish investor who buys a call might always consider the sale of an out-of-the-money call to recapture a small part of the premium paid for the low strike call. Sometimes this can be done shortly after initially entering into a long call position, particularly if the stock moves in the investor's favor. This is called "legging" into a position, and in this case the investor is legging into a bullish spread with calls. For example, one might buy the Jan 70 Home Depot calls for 7.40. A few days later, if Home Depot has risen from 68.75 to, say, 74, perhaps the Jan 75 calls might have a bid quote of 7. The investor can then "leg" into the bullish vertical spread by writing the Jan 75 calls. The net investment is then only $0.40 (the Jan 70 calls were bought for 7.40 and the Jan 75 calls were sold for 7). If Home Depot rises to 75 or higher at the expiration day the maximum time T, cash inflow of 5 will be realized.

Another argument for using vertical spreads applies to a writer of naked options, who can effectively buy insurance against large losses by buying out of the money options. For example, an investor who believes that Home Depot's stock price will rise might write the Jan 70 put to generate, and hopefully keep, its premium of, say, $700. The risk is that large losses are possible if Home Depot shares fall in price. If Home Depot Stock fell to zero, the writer of the Jan 70 put faces a loss of $6300 ($7000−$700). The purchase of the Jan 60 put represents insurance that would limit the loss to $300 plus the cost of the Jan 60 put ($700−$1000−cost of Jan 60 put).

The primary reason for avoiding vertical spreads, and indeed all the more complex strategies considered in this chapter, is that they double the commissions paid. Indeed, many have accused brokerage houses of pushing these strategies on their clients to generate additional commissions. Investors who use strategies such as spreads should deal with discount brokers, who charge just a few dollars per contract.

Finally, note that all the vertical spreads we examined were 1:1 (i.e., one option was bought and one was sold). Different payoff patterns are created by varying this ratio. Problems at the end of the chapter will allow you to create and examine these diagrams. Sometimes these strategies can be used to exploit mispriced options.

15.8 STRADDLES AND STRANGLES

Straddles and strangles are examples of "combinations," which are positions involving both puts and calls. Traders buy a straddle or strangle when they expect the underlying asset to be volatile but do not have any beliefs about whether the price will rise or fall. That is, the trader believes it will *either* rise *or* fall considerably.

A straddle is *purchased* by *buying* a put and a call on the same stock, with each option having the same exercise price and the same time to expiration. For example, from Figure 14.8, we see that one could initially buy a June 5700 call and a June 5700 put on a Swiss franc futures contract. The price of the call is $0.0105/SF, which is $1312.50 because the underlying asset is SF125,000. The price of the put is $0.0031/SF, which is $387.50. Thus, the total initial cash outflow for buying the put and the call is $CF_0 = -\$1700$.

In the following tabulations S_T is the expiration day price of a June Swiss franc futures contract, in terms of $/SF. In addition, C_T and P_T are the expiration day values of the call and the put, respectively, when the underlying asset is SF125,000; $CF_T = C_T + P_T$. The last column adds the initial day cost of the straddle (−$1700) to the expiration day cash flow.

S_T	C_T	P_T	CF_T	CF_0+CF_T
0.53	0	5000	5000	3300
0.535	0	4375	4375	2675
0.54	0	3750	3750	2050
0.545	0	3125	3125	1425
0.55	0	2500	2500	800
0.555	0	1875	1875	175
0.56	0	1250	1250	−450
0.565	0	625	625	−1075
0.57	0	0	0	−1700
0.575	625	0	625	−1075
0.58	1250	0	1250	−450
0.585	1875	0	1875	175
0.59	2500	0	2500	800

0.595	3125	0	3125	1425
0.6	3750	0	3750	2050
0.605	4375	0	4375	2675
0.61	5000	0	5000	3300

At even lower or higher prices, the straddle buyer makes greater profits. The profit diagram for the straddle buyer is shown in Figure 15.15.

A long **strangle** strategy involves the purchase of a put and a call at *two different* exercise prices. This allows an investor to position the strategy according to his beliefs and it can lower the initial cost of the strategy relative to purchasing an at-the-money straddle. For example, suppose the current futures price of a June SF futures contract is \$0.5774/SF, and an investor believes that the Swiss franc is more likely to move sharply lower than it is to increase in price. Then, purchasing the June 5800 futures call (for a premium of \$0.0053/SF; see Figure 14.8) and the June 5700 futures put (for a premium of \$0.0031/SF) would allow speculation on these beliefs. The initial cost of this strategy is $CF_0 = -\$1050$. The profit diagram of this strangle is present in Figure 15.16. You should prepare the profit table and compare it to Figure 15.16.

Figure 15.16 shows that the initial cost of the strangle strategy, which is also the maximum loss if both options expire worthless, is \$1050. This is less than the initial cost of buying a straddle. In addition, the strangle buyer makes a greater profit (or smaller loss) if $S_T < \$0.575$/SF. If the strangle buyer instead thought there was a greater probability that the Swiss franc would move higher rather than decline, he might buy the June 5700 call and buy the June 5650 put. Such a strategy shifts the strangle diagram in Figure 15.16 to the left. Problem 15.9 at the end of the chapter asks you to verify this by preparing a profit table and drawing a profit diagram.

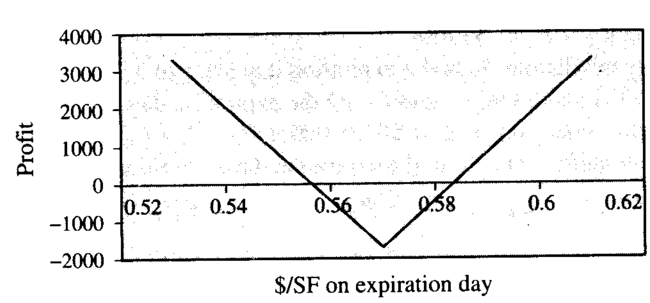

Figure 15.15 Buying a straddle.

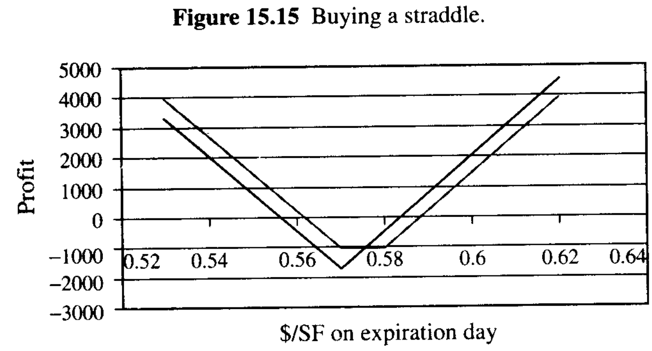

Figure 15.16 Long strangle vs long straddle.

The straddle or strangle *writer* profits from price stability. In selling a straddle, a put and a call with the same exercise price are sold. For example, see Figure 14.5, and consider the written straddle using the Cisco October 20 options:

Sell 1 Cisco October 20 call	+3.30
Sell 1 Cisco October 20 put	+3.90
Initial cash flow, CF_0	+7.20

At expiration, we have the following:

	At Time T			
S_T	**Buy Jan 20 Call**	**Buy Jan 20 Put**	**Cash Flow CF_T**	$CF_0 + CF_T$
17	0	−3	−3	+4.20
18	0	−2	−2	+5.20
19	0	−1	−1	+6.20
20	0	0	0	+7.20
21	−1	0	−1	+6.20
22	−2	0	−2	+5.20
23	−3	0	−3	+4.20

In this example, losses are realized if Cisco falls below 12.80 or rises above 27.20. The profit diagram is shown in Figure 15.17.

There are two sources of substantial risk in writing a straddle. First, unless Cisco is exactly at 20 on and just before the option's expiration day, there will always be some risk of early exercise because one of the options will always be in the money. In Chapter 16, we will learn that an in-the-money put may always be exercised early. Furthermore, while an in-the-money call *should* be exercised early only immediately before an ex-dividend day, some investors might wish to exercise a few days before the ex-date, and others might exercise a few days before the expiration date even in the absence of dividends. For example, tax considerations might motivate early exercise of an in-the-money call. Second, there is risk of significant losses if Cisco rises or falls sharply. At $S_T = 50$, for example, the total loss is \$2280. The potential losses should Cisco rise in price are theoretically unlimited.

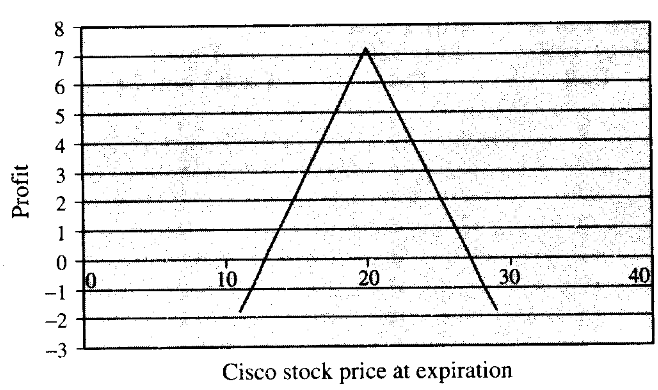

Figure 15.17 Written straddle.

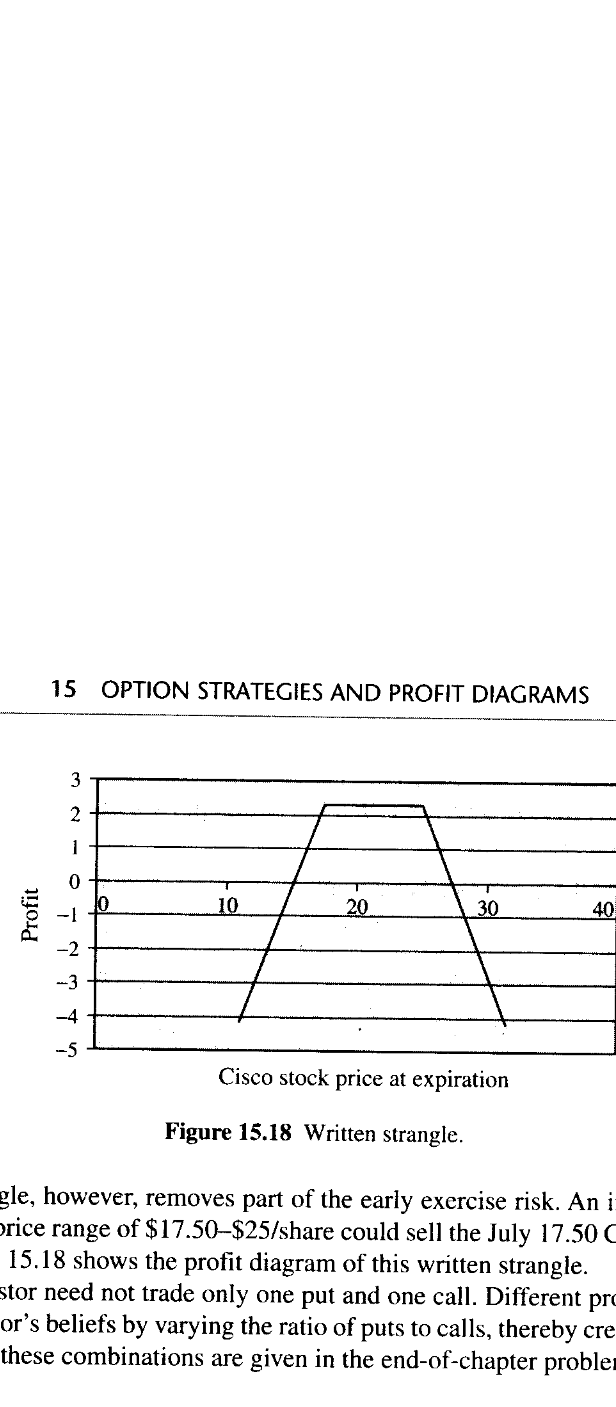

Figure 15.18 Written strangle.

Selling a strangle, however, removes part of the early exercise risk. An investor who expects Cisco to trade in a price range of $17.50–$25/share could sell the July 17.50 Cisco put and sell the July 25 call. Figure 15.18 shows the profit diagram of this written strangle.

Again, an investor need not trade only one put and one call. Different profit strategies can be tailored to an investor's beliefs by varying the ratio of puts to calls, thereby creating different profit diagrams. Some of these combinations are given in the end-of-chapter problem set.

15.9 Synthetic Forward

Buying a call and simultaneously writing a put with the same strike price and time to expiration creates a "synthetic" long forward position in the underlying asset. For example, observe the rightmost column of Figure 14.6. Suppose a trader buys the May 1250 SPX call and writes the May 1250 SPX put.

Buy 1 SPX 1250 call	−19.90
Sell 1 SPX 1250 put	+12.50
	$CF_0 = -7.40$

Thus, this strategy costs $740 to initiate. At expiration, both option positions are offset:

S_T	**Sell 1 SPX May 1250 Call**	**Buy 1 SPX May 1250 Put**	**Time T Cash Flow, CF_T**	**Profit/Loss CF_0+CF_T**
1247	0	−3	−3	−10.4
1248	0	−2	−2	−9.4
1249	0	−1	−1	−8.4
1250	0	0	0	−7.4
1251	+1	0	+1	−6.4
1252	+2	0	+2	−5.4
1253	+3	0	+3	−4.4
1254	+4	0	+4	−3.4
1255	+5	0	+5	−2.4
1256	+6	0	+6	−1.4

1257	+7	0	+7	−0.4
1258	+8	0	+8	+0.6
1259	+9	0	+9	+1.6

The profit diagram is shown in Figure 15.19.

Note that Figure 15.19 is similar to the long stock position shown in Figure 15.1. However, there are some differences between actually buying the underlying asset (SPX) and taking the long synthetic forward position. First, consider that one must invest $125,518 to effectively buy 100 shares of the S&P 500 (the spot SPX was at 1255.18 on May 10, 2001), but only $740 to assume the long synthetic forward position. Clearly, the return on investment will be higher for the synthetic forward. Second, while the breakeven price is $1255.18 if the underlying spot asset is purchased, the breakeven price is $1257.40 for the synthetic forward (the strike price plus the cost of the position). This synthetic forward price of $1257.40 should theoretically equal the cost-of-carry pricing model price ($F = S +$ carry costs − future value of dividends); see Chapter 5. Finally, be aware that selling the May 1250 put will entail a substantial initial margin requirement.

Symmetrically, writing a call and buying a put creates a synthetic *short* forward position in the underlying asset. Using the same May 1250 SPX options, this strategy generates a cash inflow of $740 at initiation:

Sell 1 SPX 1250 call	+19.90
Buy 1 SPX 1250 put	−12.50
	$CF_0 = +7.40$

At expiration, both option positions are offset:

S_T	Buy 1 SPX May 1250 Call	Sell 1 SPX May 1250 Put	Time T Cash Flow, CF_T	Profit/Loss $CF_0 + CF_T$
1247	0	+3	+3	+10.4
1248	0	+2	+2	+9.4
1249	0	+1	+1	+8.4
1250	0	0	0	+7.4
1251	−1	0	−1	+6.4
1252	−2	0	−2	+5.4

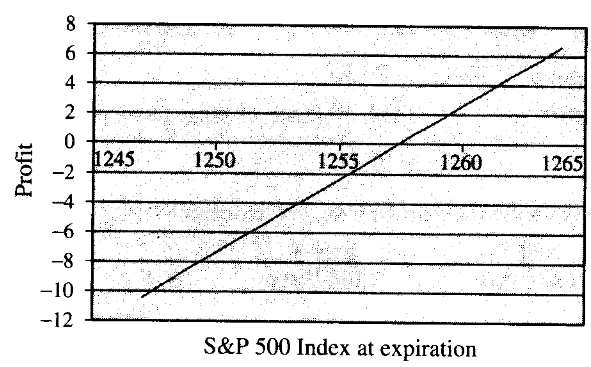

Figure 15.19 Synthetic long forward position in the SPX.

Figure 15.20 Synthetic short forward position in the SPX.

1253	-3	0	-3	+4.4
1254	-4	0	-4	+3.4
1255	-5	0	-5	+2.4
1256	-6	0	-6	+1.4
1257	-7	0	-7	+0.4
1258	-8	0	-8	-0.6
1259	-9	0	-9	-1.6

The profit diagram from this synthetic short forward strategy is shown in Figure 15.20. Note that the trader is effectively short at a price of $1257.40 (the strike price plus the premium generated at initiation).

15.10 Other Strategies

By now you should be able to draw any profit diagram for any strategy. For example, consider a put ratio spread. A put ratio spread consists of one long put with a relatively high strike price (K_H), and selling two, three, or more puts with a relatively low strike price (K_L). Let's use the SPX price data in Figure 14.6 to construct a put ratio spread profit table and profit diagram for June put options (they are in the right-most column of Figure 14.6). Initially, we buy one June 1325 put and sell three June 1300 puts. Thus, $K_H = 1325$ and $K_L = 1300$. The time 0 cash flows are as follows:

Buy 1 June 1325 put	-74.50
Sell 3 June 1300 puts	+166.50 (each of these has a premium of 55.50)
	$CF_0 = +92.00$

The time T cash flows, and the total profit/loss, are as follows:

S_T	**Sell 1 1325 Put**	**Buy 3 1300 Puts**	CF_T	$CF_0 + CF_T$
1200	125	-300	-175	-83
1205	120	-285	-165	-73
1210	115	-270	-155	-63

1215	110	-255	-145	-53
1220	105	-240	-135	-43
1225	100	-225	-125	-33
1230	95	-210	-115	-23
1235	90	-195	-105	-13
1240	85	-180	-95	-3
1245	80	-165	-85	7
1250	75	-150	-75	17
1255	70	-135	-65	27
1260	65	-120	-55	37
1265	60	-105	-45	47
1270	55	-90	-35	57
1275	50	-75	-25	67
1280	45	-60	-15	77
1285	40	-45	-5	87
1290	35	-30	5	97
1295	30	-15	15	107
1300	25	0	25	117
1305	20	0	20	112
1310	15	0	15	107
1315	10	0	10	102
1320	5	0	5	97
1325	0	0	0	92
1330	0	0	0	92
1335	0	0	0	92
1340	0	0	0	92
1345	0	0	0	92
1350	0	0	0	92

The resulting profit diagram is shown in Figure 15.21.

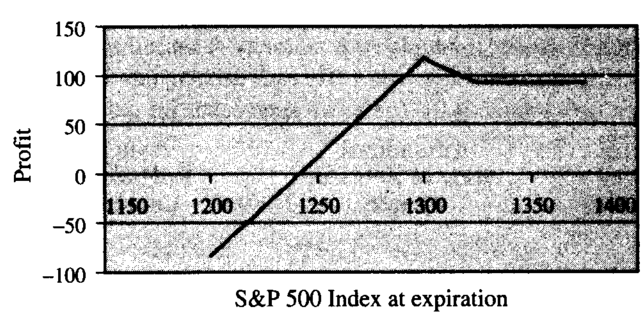

Figure 15.21 Put ratio spread.

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The maximum profit is +\$11,700, which is realized if $S_T = 1300$. At all $S_T \geq 1325$, a profit of \$9200 is realized. This is represented by the horizontal line segment to the right of $K_H = 1325$. Note that the slope of the line to the left of $K_L = 1300$ is 2. That is, for every dollar below $S_T = 1300$, the strategy loses two additional dollars. This bullish strategy breaks even at $S_T = 1241.50$.

15.11 RATE-OF-RETURN DIAGRAMS

In a rate-of-return diagram, the rate of return on an investor's initial outlay is on the y axis, and S_T is on the x axis. You should prepare rate-of-return diagrams only if there is an initial cash outflow.[12]

EXAMPLE 15.1 Consider an investor who writes a covered call. The initial parameters are $S_0 = 38$, $K = 40$, $C_0 = 3$, and $T =$ three months. Thus, the initial cash flows are as follows:

Buy 100 shares of stock	−38 (per share)
Write 1 call	+3 (per share)
Initial cash flow at time 0	−35 (per share)

At expiration, we have the following results:

Sell Stock for S_T	Buy Call for C_T	Total Cash Flow at Time T	Profit/Loss $CF_0 + CF_T$	Rate of Return
+34	0	+34	−1	−2.86%
+35	0	+35	0	0%
+36	0	+36	+1	+2.86%
+37	0	+37	+2	+5.71%
+38	0	+38	+3	+8.57%
+39	0	+39	+4	+11.43%
+40	0	+40	+5	+14.29%
+41	−1	+40	+5	+14.29%
+42	−2	+40	+5	+14.29%

The profit diagram is depicted in Figure 15.22a, and the rate-of-return diagram in Figure 15.22b. Note that the rate of return is over a three-month holding period. It can be annualized in one of two ways: simple interest (multiply each return by 4, since there are four 3-month periods in a year), or compounded interest (add 1.0 to each decimal return and take the number to the fourth power, then subtract 1.0). These two approaches are illustrated as follows:

Three Month Unannualized Rate of Return	Annualized Rate of Return, Simple	Annualized Rate of Return, Compounded
−2.86%	$(-.0286)(4) = -11.43\%$	$(0.9714)^4 - 1 = -10.95\%$
0%	0%	0%
+2.86%	+11.43%	+11.93%

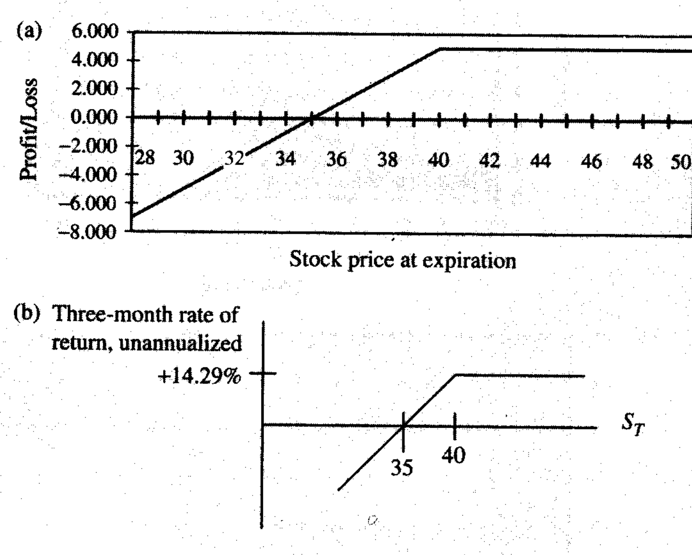

Figure 15.22 Writing a covered call: (*a*) profit diagram and (*b*) rate-of-return diagram.

+5.71%	+22.86%	+24.89%
+8.57%	+34.29%	+38.95%
+11.43%	+45.71%	+54.17%
+14.29%	(14.29)(4)=+57.14%	$(1.1429)^4-1$=+70.60%

Also note that the unannualized rate of return of 14.29% is achieved if the stock price rises from 38 to as little as 40. In contrast, the unannualized rate of return on the stock if $S_0=38$ and $S_T=40$ is only (40−38)/38=5.263%.

15.12 Profit Diagrams for Different Holding Periods

Investors do not have to hold their positions until expiration. Profit diagrams for shorter holding periods can be created, though you must *estimate* what each option will be worth at different times to expiration. Option pricing models are used to estimate future option values, but they will not be presented until Chapters 17 and 18. FinCAD provides two option strategy simulators that illustrate the "decay process" of option strategies. Click on FinCADXL, and look under the "Options–vanilla" menu. The two choices for illustrating strategy decay diagrams are Option Strategy Simulator–New and Option Strategy Simulation–Two Options. Figure 15.23a shows how the new Option Strategy Simulator diagrams the time-evolving profits and losses from a bull vertical spread. The price of the underlying asset on the day that the strategy is first created is 68.75. Figure 15.23b depicts the same "decay process" for a purchased straddle. Figure 15.23c gives the values that were entered into the "option details" portion of the FinCAD spreadsheet for the straddle. The put and the call were purchased on the start date. The options expire on the expiry date. The value date is the date on which the profits and losses are computed. The decay of a strategy shows how profits and losses change at different stock prices as time passes, all else equal.

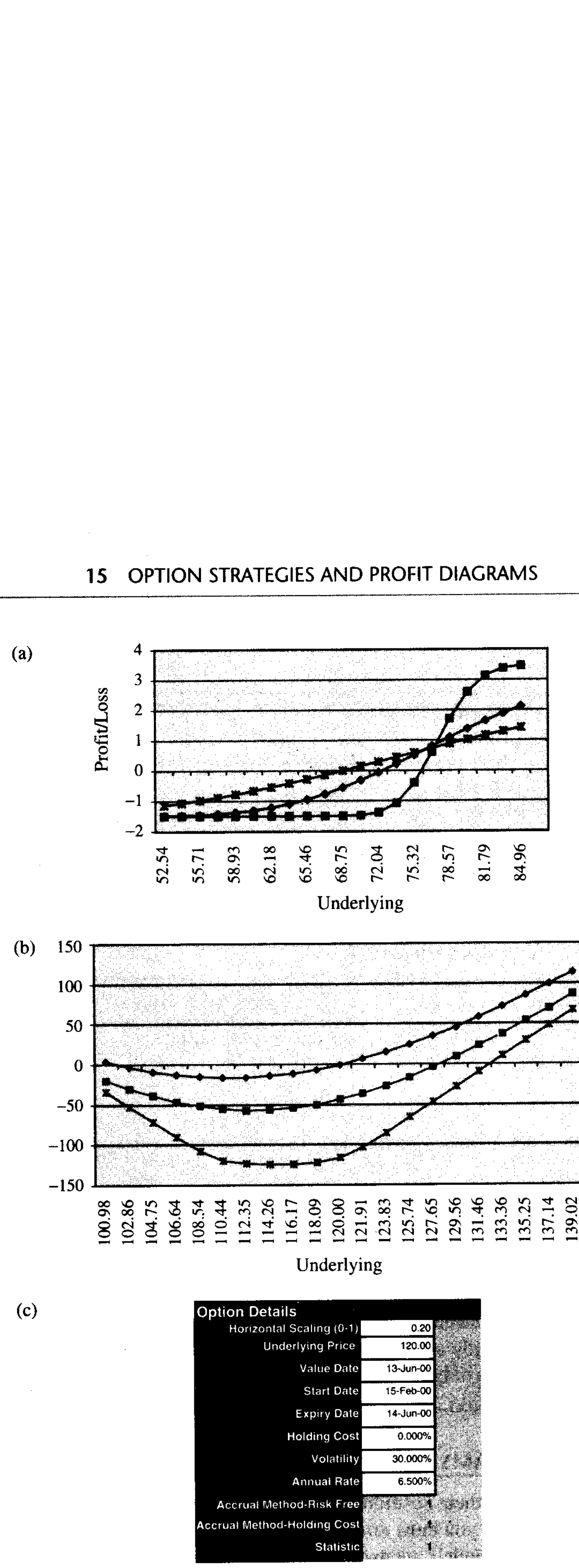

Figure 15.23 Decay diagrams from FinCAD's Option Strategy simulator: (*a*) for a bullish money spread with calls, and (*b*) for a purchased straddle. (*c*) Values entered as "option details" for the straddle. Curves show profit and loss as follows: asterisks, one day later (assuming no change in the implied volatility); diamonds, one month later; squares, seven weeks later (i.e., one day before options expire).

15.13 Several Caveats

Before using the strategies described in the preceding sections, you must be awáre of several factors that will affect the profit or loss ultimately realized.

a. First, there is no reason that these positions must be held until expiration. They can be closed out early, and doing so will affect the nature of the profit diagram. For example,

EXAMPLE 15.2 Today is June 2. July 155 puts are priced at $3\,^1/_8$. July 150 puts have a premium of $1^1/_2$. For an investor who wishes to create a bullish vertical spread with puts, the initial cash flows on June 2 are as follows:

Sell 1 July 155 put	$+3\,^1/_8$
Buy 1 July 150 put	$-1\,^1/_2$
Initial cash flow, CF_0	$+1\,^5/_8$

Now, suppose that the investor wishes to observe the profit diagram as it would look if the strategy was closed out on June 25, which is 22 days before the expiration date of July 17. The values of the puts must be estimated for a range of stock prices, and assuming that the puts have 22 days remaining to expiration. It is possible that the June 25 cash flow table would be as follows[13]:

$S(T)$	Buy One July 155 Put	Sell One July 150 Put	Cash Flow on June 25	Total Cash Flow ($+1\,^5/_8$+June 25 Cash Flow)
143	−9.2131	6.4931	−2.7201	−1.0951
145	−7.5760	5.1348	−2.4412	−0.8162
147	−6.0458	3.9155	−2.1303	−0.5053
149	−4.6361	2.8408	−1.7953	−0.1703
151	−3.3576	1.9110	−1.4466	0.1784
153	−2.2172	1.1221	−1.0951	0.5299
155	−1.2175	0.4658	−0.7517	0.8733
157	−0.3567	−0.0695	−0.4262	1.1988
159	0.3710	−0.4974	−0.1263	1.4987
161	0.9750	−0.8327	0.1423	1.7673
163	1.4670	−1.0904	0.3765	2.0015
165	1.8603	−1.2846	0.5757	2.2007
167	2.1689	−1.4281	0.7408	2.3658
169	2.4067	−1.5322	0.8745	2.4995
171	2.5866	−1.6064	0.9803	2.6053
173	2.7203	−1.6582	1.0621	2.6871
175	2.8179	−1.6938	1.1241	2.7491
177	2.8880	−1.7179	1.1701	2.7951
179	2.9374	−1.7339	1.2036	2.8286
181	2.9717	−1.7443	1.2274	2.8524
183	2.9951	−1.7510	1.2441	2.8691

Figure 15.24 shows the June 25 cash flows (not profits) to this strategy, assuming that all positions are held only until June 25th.

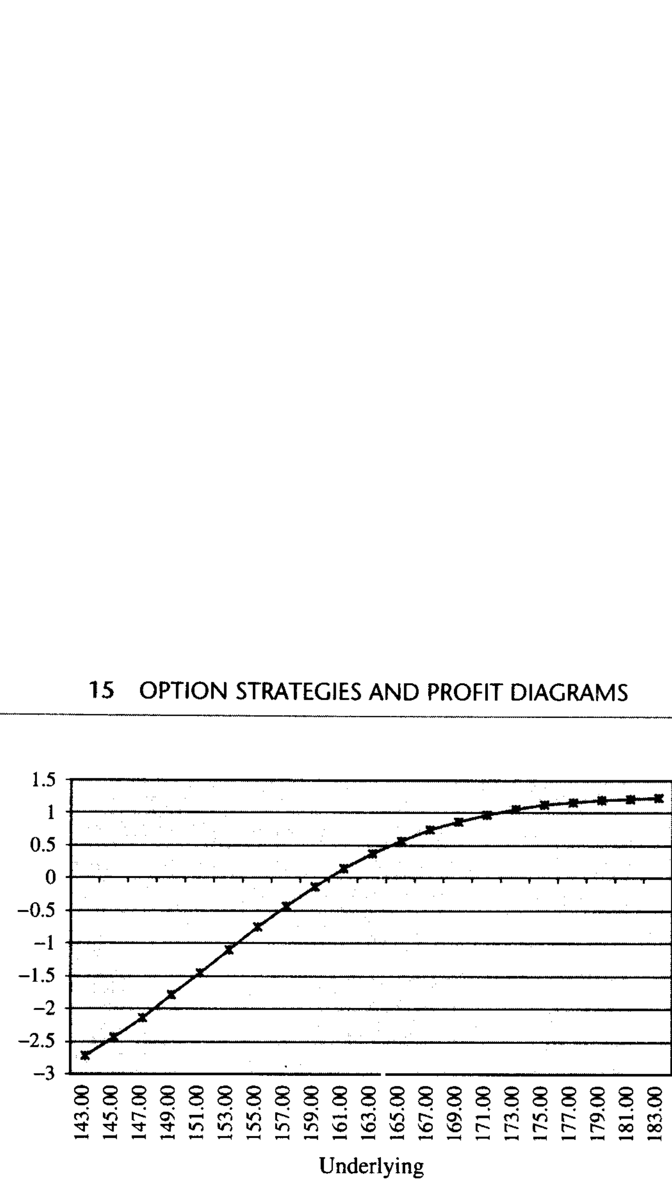

Figure 15.24 A FinCAD Option Strategy simulator diagram for *payoffs* of bullish vertical spread created in Example 15.2.

Figures 15.23a and 15.23b show how the profit diagrams for a bullish vertical spread and purchased straddle might change if they were closed out prior to expiration.

b. Keep in mind that bid–ask spreads exist for options. Thus, an investor buys at a higher asked price and sells at a lower bid price. Bid–ask spreads are a cost of trading and will reduce profits (or increase losses). Sometimes, an investor will have to pay more than $S_T - K$ for a call purchased at expiration, or receive less than that amount if sold, because the bid price is below the call's intrinsic value and the asked price is above it. Indeed, when the expiration day is near, bid prices of deep in-the-money calls and puts are frequently less than $S - K$ and $K - S$, respectively. Floor traders arbitrage under these conditions.

c. Brokerage commissions can be substantial.

d. We have ignored dividend payments on the stock.

e. Perhaps most important, we have ignored the risk of early exercise. Most of the options traded in the United States are American options. As such, a written put that is in the money might be exercised early at any time. Written in-the-money calls are most likely to be exercised just prior to an ex-dividend date, though transactions costs and tax considerations might induce the in-the-money call owner to exercise even when there are no dividends. We will cover the topic of early exercise in greater detail in Chapter 16. Keep in mind, however, that early exercise can add to the investor's transactions costs, margin requirements, and risk.

f. We have ignored the margin requirements required for written option positions.

g. We have ignored the timing of the cash flows, and the time value of money.

h. Most of the examples in this chapter had 100 shares of stock as the underlying asset. The profit diagrams from these strategies can also be realized when one is trading foreign exchange options and futures options.

15.14 Research on Option Strategies

Several empirical papers have attempted to measure the risks and rewards to investors who use some of the strategies discussed in this chapter. When reading any of these, keep in mind the caveats of Section 15.13. Many of the early papers used closing daily option prices and did not account for market frictions such as transactions costs, bid–asked spreads, and nonsynchronous trading. Also, the realized results of any strategy can be time specific. For example, writing covered calls will provide superior results in a neutral market but will underperform many strategies when the market rises sharply. Be cautious in interpreting strategy results obtained over a short time period. Black (1975) is an insightful early article on using options.

Another important fact to remember is this:

> If options are correctly priced, no option strategy should offer superior returns, given its level of risk. The different strategies should appeal to different investors based on their beliefs and on their levels of risk aversion.

A related issue entails evaluating the performance of portfolio managers when they use options. Options create highly skewed returns distributions, so that the usual method of comparing the expected return to a risk measure such as the variance of the returns distribution is not advised. See Bookstaber and Clarke (1984), Galai and Geske (1984), and Ferguson (1987) for some of the issues in performance evaluation when portfolio managers use options.

15.15 Summary

In this chapter, we demonstrate how profit tables and profit diagrams can be prepared for any simple or complex option strategy. There are six elementary strategies: long and short stock, buy and write a call, and buy and write a put. All complex strategies are combinations of these six basic positions.

When option positions are held until expiration, the payoff patterns differ considerably from those that can be achieved by using only the underlying security. Thus, the payoffs can be patterned according to an investor's beliefs and level of risk aversion. Indeed, this is often cited as an economic rationale for the existence of options.

In Chapter 17, however, we will see that over shorter periods of time (e.g., one day), and for a specified probability distribution of stock price movements, a call option is equivalent to a levered position in the stock. In other words, the payoff pattern of a long call position will be seen to be identical to the one provided by a long position in the stock financed in part by borrowed funds. A long put is equivalent to the short sale of stock and lending. In the absence of transactions costs, the payoffs provided by an option can be replicated by adjusting the investment in stocks and bonds as time passes, and as the price of the stock changes. Therefore, realizing the payoff patterns of the strategies in this chapter should not necessarily be cited as a reason for the existence of options. However, one reason that options exist is reduced transactions costs.

McMillan (1992) provides additional details on the different strategies covered in this chapter.

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Notes

[1]Recall that the subject of short selling is covered in detail in Appendix A of Chapter 5.

[2]An investor who is very bearish should buy puts.

[3]An investor who is very bullish should buy calls. The writer of an at-the-money naked put, or an out-of-the-money naked put, can be neutral on the stock.

[4]If the AMAT put was only slightly in the money at expiration (e.g., $S_T = 49.80$), a put owner might not exercise because transactions costs might exceed the in-the-money amount. Some brokers will automatically exercise puts that are in the money by some minimum amount on the expiration day.

[5]In a horizontal spread, the investor typically writes a call with a short time to expiration and buys a call with a longer time to expiration. The same can be done with puts.

[6]The covered call writer would likely be assigned the written call by the call holder. In this case, he receives the strike price of 20 upon delivery of the shares. The profit diagram is the same as the example in the body of the text. In practice, when deciding whether to close an in-the-money option position early, a call writer should consider commissions, dividends, whether he wishes to continue owning a particular stock (here, Cisco), and his subjective perceived probability that this stock will decline in price before expiration.

[7]The six basic positions are long stock, short stock, long call, short (written) call, long put, and short put. All the more complex strategies consist of combinations of the basic positions.

[8]See Brown and Lummer 1986, 1988, and Zivney and Alderson (1986).

[9]The basic put–call parity theorem is $C - P = S - PV(K)$. In other words, the difference between a call and a put on the same stock with the same strike price and time to expiration equals the value of the stock less the present value of K. Rearranging, $(-S+C) - P = -PV(K)$, which says that the difference between the cost of a covered call $(-S+C)$ and the cost of a long position in a put $(-P)$ equals the investment of $\$PV(K)$.

[10]If another option with a different strike price and a different time to expiration were to be sold, it would be called a **diagonal spread**.

[11]As will be discussed in Chapter 16, if in-the-money calls will ever be exercised early, it will be optimal to do so only on the day before the stock trades ex-dividend.

[12]Rate-of-return diagrams are tricky to interpret when the initial cash flow is positive (i.e., an initial cash inflow). You might, however, consider the interest rate you compute to be a borrowing cost, because you are effectively borrowing money when you initially have a cash inflow. Then, your goal is to borrow that money at the lowest possible rate.

[13]The put values were estimated using the Black–Scholes put option pricing model, with no dividends. The volatility is 30%, and the riskless interest rate is 6.5%. See Section 18.10.1 of Chapter 18 for discussion of this model.

PROBLEMS

15.1 You buy a call for 2 5/8. The strike price is 60. What is the maximum loss you can possibly realize? What is the maximum profit? At what expiration day stock price will you break even?

15.2 You write a naked call. The call premium is 4 3/4. The strike price is 40. What is the maximum loss you can possibly realize? What is the maximum profit? At what expiration day stock price will you break even?

15.3 Refer to Figure 14.5. Suppose you buy the Broadcom June 40 put option. Prepare the profit table and profit diagram. What is the breakeven expiration day stock price?

15.4 Refer to Figure 14.5. Suppose you sell the AmOnline (America Online) July 50 put option. Prepare the profit table and profit diagram. What is the breakeven expiration day stock price? Why would an investor write this put? What are the benefits and costs of writing this naked put compared with placing a limit order to buy the stock at a price of $50/share?

15.5 Figure 15.8 uses Cisco common stock to illustrate a covered call strategy. It shows that the losses from this strategy can be significant if the stock price declines sharply. These losses can be capped by purchasing a put. Using the data from Figure 14.5, prepare a profit table and a profit diagram for a strategy in which you buy 100 shares of Cisco, sell one July 20 call, and buy one July 15 put.

15.6 An investor wishes to buy a bullish spread using calls on eBay. Refer to Figure 14.5. Construct the profit diagram if the May 50 and May 55 calls are used. Then construct the profit diagram if the May 55 and May 60 calls are used. Compare the two diagrams. Discuss the advantages and drawbacks of each.

15.7 Define C_H as the premium of a call with a high strike price, K_H. Define C_L as the premium of a call with a low strike price, K_L. What is the maximum profit that can be realized from a bullish spread with calls? What is the maximum loss from a bullish spread with calls?

15.8 Prepare the profit table for the strangle strategy shown in Figure 15.16.

15.9 Prepare a profit table and a profit diagram for a strangle strategy in which the Swiss franc June 5700 call and the June 5650 put are purchased. Compare the result with that for the strangle strategy in Figure 15.16. How is it different?

15.10 Use the data from Figure 14.5 to prepare a profit table and a profit diagram sketch for the following strategy, assuming that all positions are held until expiration. The underlying security is General Electric (Gen El).

Sell short 200 shares

Buy 2 June 50 calls

Sell 1 June 50 put

Sell 1 June 55 call

15.11 Some floor traders will "leg into two ratio spreads." This strategy is used when a great deal of near term volatility is expected, followed by a large price increase or a large price decrease. For example, suppose a floor trader believed that Qualcomm was going to announce a surprise earnings figure at the end of June 2000, at which time the stock would move to either $180 or $140 per share. Until then, the trader believed that the stock would likely exhibit considerable volatility between

$155 and $165 per share. Reflecting this uncertainty, he observed that the July 160 puts and calls were overpriced relative to their "normal" values. Assume the July 160 put sold for 7 and the July 160 call sold for 8 on June 3, 2000.

The investor could have bought the high-priced puts and calls (a straddle), thereby profiting if his beliefs were correct. To realize a profit, the stock would then have had to move 15 points either up or down, from its current value of $159.375 share.

Instead, the trader could have legged into two ratio spreads. Legging into a position involves piecemeal trades over time, with the risk that one leg (or more) might not get executed. First, he could have sold one July 160 put (for 7) and one July 160 call (for 8). At this stage, if Qualcomm's price had remained stable (i.e., the trader was wrong), he likely would have profited as the time value of the July 160 options decayed.

Suppose instead, that two days later, Qualcomm had made a move to 165, such that one leg of the back spread would be entered by buying two out-of-the-money July 155 puts. Assume that at this time the July 155 puts sold for 3 1/8 each. Now, the trader's belief of considerable short-term volatility would be tested. Assume that the trader was correct and that four days after the July 155 puts were bought, Qualcomm indeed fell to 155. The last leg of the back spread would be entered by buying two out-of-the-money July 165 calls. Assume that they sell for 3 3/4 each.

Prepare the profit table and profit diagram for this strategy.

Note that it is unlikely that individual investors could ever profitably employ this strategy because they must buy the options at the ask price and sell at the bid. They also must pay high commissions on the options. Finally, be cognizant of the great risk that exists until all the legs have been entered.

15.12 Suppose that on an airplane flight you overhear one businessman talking excitedly to another about Dell. You also think that you hear the terms "financial distress" and "takeover candidate." Finally, you hear one of the businessmen say, "Dell's stock price will really move when this becomes public." However, because the pilot rudely interrupted your eavesdropping, you do not know whether your unwitting informant expects Dell's stock price to move up or move down. Suppose that the date of the flight is May 10, 2001, which also happens to be the date of the price data in Figure 14.5. Discuss the options trades you would make based on your information.

15.13 Using the July 20 and July 25 puts and calls for Cisco in Figure 14.5, construct a long "box spread." That is, prepare a profit table and a profit diagram for the following positions:

Buy 1 July 20 call

Sell 1 July 20 put

Sell 1 July 25 call

Buy 1 July 25 put

What do you notice about this strategy? Suppose you *reverse* the positions in each of the four options. What do you notice now about this strategy?

Use FinCAD's "Option Strategy Simulator–New" to analyze how this strategy changes over time, all else equal.

15.14 Use call data from Figure 14.5 for AmOnline to construct a "long butterfly." That is, prepare a profit table and a profit diagram for the following positions:

Buy 1 July 50 call

Sell 2 July 55 calls

Buy 1 July 60 call

Use FinCAD's "Option Strategy Simulator–New" to analyze how this strategy changes over time, all else equal.

Now, suppose you had read the data from the wrong column and used puts instead of calls. Redo the profit table and profit diagram using the put data. What do you notice?

Finally, reverse the positions in each call option and redo the profit table and profit diagram for this "short butterfly."

15.15 Use call data from Figure 14.6 for S&P 500 (SPX) options to construct a "long condor." That is, prepare a profit table and a profit diagram for the following positions:

Buy 1 May 1250 call

Sell 1 May 1275 call

Sell 1 May 1300 call

Buy 1 May 1325 call

Use FinCAD's "Option Strategy Simulator–New" to analyze how this strategy changes over time, all else equal.

Suppose you had read the data from the wrong column and used puts instead of calls (Duh!). Redo the profit table and profit diagram using the put data. What do you notice?

Finally, reverse the positions in each of the four case options, (this is known as a "short condor.") Redo the profit table and profit diagram.

15.16 A stop-loss order works like this. You tell your broker to sell some stock that you own if its price ever declines to some lower level. If the stock reaches that level, your stop-loss order will automatically become a market order to sell (or a limit order to sell). Compare the benefits and costs of these two alternative strategies: (a) buy stock and place a stop-loss order at a price $5 below the current stock price and (b) buy an in-the-money call option, with the strike price $5 below the current stock price.

15.17 Fill in the two blanks in the following, which is based on a *Wall Street Journal* article dealing with LEAPS on a different stock.

> The new long-term options could also be used to protect profits on stock bought earlier. For example, suppose an investor owned 100 shares of America Online purchased several years ago at $65 a share. The stock is currently selling for around $144. Suppose the nervous investor is worried about losing some of the gain if the market goes into the soup. To sleep better, the investor could try a "hedge wrapper." To construct a hedge wrapper, sell a call with a strike price above the current price of the stock and purchase a put with a strike price below the stock's price. In the AOL case, the worried investor could sell a call with a strike price of $155, while buying a put with a $130 strike price. Both options expire in January 2000. The investor would receive a brokerage credit of $1012.50 for selling the call and would pay $725 for the put—a net credit of $287.50, not counting commissions.
>
> This strategy would not only offer protection against a stock price slide, it would ensure a profit—again, not counting commissions—of at least ___ % on the combined stock and stock options positions. The investor's downslide protection comes from the put, which gains in value if the stock price drops below the put's strike price.
>
> The trade-off is that the return would be capped at ___%, no matter how high the stock rallied. That's because the investor could expect to have the stock "called" away at the $155 strike price once AOL's stock price rose above $155 a share.

15.18 In the late 1980s, two companies, Marion Merrill Dow and Rhône-Poulenc, issued **contingent value rights** (CVRs) that traded on the American Stock Exchange. To describe a CVR, two useful definitions are as follows:

BP = base price

TP = target price

If S_T is the stock price at expiration, then a CVR pays off as follows at time T:

$0 \quad \text{if } S_T \geq TP$

$TP - S_T \quad \text{if } BP \leq S_T \leq TP$

$TP - BP \quad \text{if } S_T \leq BP$

Here are the relevant parameters, assuming that the companies do not elect to extend the expiration date (which they could do):

	BP	TP	T
Marion Merrill Dow	\$30.00	\$45.77	September 30, 1991
Rhône-Poulenc	\$52.00	\$98.26	July 31, 1993

On January 10, 1991, the Rhône-Poulenc CVR sold for \$8.25, and the common stock sold for \$67.75. Prepare a profit table and profit diagram for the Rhône-Poulenc CVR. What other strategy offers the same type of payoff structure as a CVR?

Also prepare a profit table and profit diagram for a strategy in which one share of Rhône-Poulenc and one CVR are bought. What rate of return would an investor earn if Rhône-Poulenc's stock price on July 31, 1993 was unchanged from what it was on January 10, 1991? What are the rates of return if $S_T = BP$ and if $S_T = TP$? What is the breakeven point? What is the minimum rate of return this strategy can realize?

15.19 Use FinCAD's "Option Strategy Simulator–New" template to analyze how a written strangle decays over time.

15.20 You buy a call option with a strike price of 40. The price of the call is 3. The price (today) of the underlying asset is 39. At what price would the underlying asset have to sell (S_T) for you to break even?

- **a.** 36
- **b.** 37
- **c.** 40
- **d.** 42
- **e.** 43

15.21 You write a put with a strike price of 40. The price of the put is 3. The price of the underlying asset is 39. At what price would the underlying asset have to sell (S_T) for you to break even?

- **a.** 36
- **b.** 37
- **c.** 40
- **d.** 42
- **e.** 43

15.22 The profit diagram for writing a covered call looks like

- **a.** Buying a call
- **b.** Writing a naked call
- **c.** Buying a put
- **d.** Writing a naked put
- **e.** Buying the underlying asset

CHAPTER 16

Arbitrage Restrictions on Option Prices

In Chapter 15, we presented option strategies based on the value of options at expiration. In Chapters 17 and 18, we will discuss two particularly important models that can be used to calculate option prices before the option expiration date.

However, before those two specific models are presented, it is important to realize that basic option values before expiration must obey a set of pricing restrictions, regardless of the specific option-pricing model employed. Therefore, in this chapter, we present the upper and lower boundary prices for call and put options.

We close this chapter with a particularly important relationship is known as *put-call parity*. Note that this important relationship, the subject of Section 16.4, can be studied independently of the other pricing restrictions.

To begin, recall that a fundamental assertion of modern finance is that arbitrage cannot persist. Arbitrage is a trade, or a set of trades, that produces a positive cash flow at one or more dates and zero cash flows at all other dates. In well-functioning markets, options, like any other security, cannot be priced such that arbitrage opportunities exist. If the chance to realize a riskless arbitrage profit were to appear, investors would immediately move to exploit it, and in the process, prices would correct themselves. This chapter also establishes important concepts concerning the early exercise decision for American options on equities.[1]

We make the following assumptions in deriving the arbitrage restrictions described in this chapter:

a. No commissions or transactions costs.

b. All trades take place at a single price (i.e., there is no bid–ask spread, and trading does not affect the market price).

c. No taxes.

d. No margin requirements and no short sale restrictions. Investors receive the full amount of written options and shares sold short.[2]

e. Investors can trade in the stock and options markets instantaneously. Investors get immediate notice of option assignments.

f. Dividends are received on the ex-dividend day, and the ex-day stock price decline equals the dividend amount.

g. Investors prefer more wealth to less, and they are quick to take advantage of any arbitrage opportunities that appear.

h. The underlying asset is 100 shares of common stock, or an index option. (Some of the results do not apply to futures options or options on foreign exchange.)

If an option or options appear to violate any of the following derived restrictions, do not believe that *you* can exploit the apparent mispricing. The foregoing assumptions probably do not hold for you unless you are a low cost trader (i.e., an institutional trader or an individual with substantial capital who has seats on the appropriate options and stock exchanges). Floor traders also have the advantage of buying at the bid price and selling at the ask price. Finally, even if you observe that an option's *closing* price appears to be violating a pricing restriction, remember that prices in the financial press are always suspect owing to nonsynchronous trading and bid–ask spreads.

16.1 NOTATION

16.1.1 Time

Some notation must be introduced. Today is defined as time zero (0). The expiration date is time T. Any intermediate dates are denoted $t1$, $t2$, etc., with $t1$ the closest to time 0. Figure 16.1 depicts a time line that should make these concepts easier to picture. At time 0, the time left until expiration equals T. At time $t1$, the time left until expiration equals $T-t1$.

16.1.2 Interest Rates

The present value, as of today, of $Z to be received at time T is $Z(1+r)^{-T}=Z/(1+r)^T$. Thus if $T=1$ year, $K=\$50$, and the yearly interest rate is r is 8%, the present value is $50/(1.08)^1=\$46.2963$. An annual 8% interest rate can be converted to a rate applicable for a shorter period of time by using either the simple interest method (so that an annual rate of 8% is a 4% semiannual rate and a 2% quarterly rate) or the compound interest method. With the latter, one solves for the periodic rate that, if compounded, yields 8% per year. That is, in this example, the semiannual rate is 3.923% (because $1.03923^2-1=8\%$) and the quarterly rate is 1.943% (because $1.01943^4-1=8\%$).

You might ask whether the interest rate is a borrowing rate or a lending rate. Under the assumptions in this chapter, the two rates are equal. However, if you were ever to try to arbitrage, use your own borrowing rate if the trades call for borrowing and your own lending rate if they require lending. If markets are functioning properly, the pricing restrictions will incorporate the borrowing and lending rates of the most advantageously situated investors (the one with the lowest borrowing rate and the one with the highest riskless lending rate).

16.1.3 Dividends

Although we assume that corporations pay dividends on the ex-dividend day, that rarely happens in reality. On average, firms actually pay their dividends about a month after the ex-day. The error

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Date	0	$t1$	$t2$	T
Time to expiration	T	$T-t1$	$T-t2$	0

Figure 16.1

created by assuming that a dividend payment of \$0.50 is made on the ex-day, rather than on the actual payment day, equals the interest that could be earned on that amount between the ex-day and payment day. If the annual interest rate is 6%, this is about $\$0.50 \times 0.06 \times {}^{1}/_{12} = \$0.0025$. The error of a quarter of a cent per share can be safely ignored.

Announcements of the ex-date and the dividend amount are usually made well in advance of the ex-date. Thus, dividends can be frequently regarded as certain, or riskless, particularly when we are dealing with short-term options. In some cases, however, there is significant probability that a dividend will be raised, lowered, or eliminated altogether. Moreover, the dividends associated with long-term options such as LEAPS should be treated as random variables.

When dividends are uncertain, we must have some new notation. Define the maximum dividend amount expected as $\overline{D}$. When we discuss the present value of $\overline{D}$, we mean the maximum present value of the largest possible dividend. Therefore, you must assess (a) the closest date you believe that the stock will trade ex-dividend (a closer date increases the present value) and (b) the largest dividend you believe the company will pay. The latter will frequently equal the most recent quarterly dividend, with some added amount to account for a possible dividend increase.

Similarly, define the smallest dividend amount you believe possible as $\underline{D}$. If you believe there is some probability that the company will actually skip its dividend payment, then $\underline{D}$ equals zero. Most of the time, $\underline{D}$ will equal the current quarterly dividend. This is because dividends are infrequently reduced or suspended. When discussing the present value of $\underline{D}$, we mean the minimum present value of the smallest dividend believed possible, which is $\underline{D}$, paid at the latest date you believe possible.

For the remainder of the chapter, we generally follow the following format. First, a proposition is made and then the proposition is proven. Then, the proposition is intuitively discussed and examples are frequently given. Moreover, each proof follows the same format. Following the proposition, we ask, "What if the proposition is violated?" If it is violated, an inequality in the form $A > 0$ is created, where A defines certain trades that will lead to a cash inflow today. Then, we will demonstrate that no matter what happens in the future, only nonnegative cash flows will be realized. The proof is then complete because arbitrage profits are earned if the proposition is violated.

The arbitrage restrictions create the smallest distance possible between the highest permitted option value and the lowest permitted option value. To say that an option's lowest value (lower bound) is zero and that its highest value (upper bound) is infinity does not provide us with much information. If we can show that an option must sell between 4 and 7, or better still, between 5.2 and 5.7, then we have better information. At the limit, the upper and lower bounds will converge, in which case we can say we have an option pricing model. Option pricing models are the subject of the next two chapters.

We are now ready to establish pricing restrictions for option prices. In Section 16.2 we set boundary conditions for calls. Upper bounds, or maximum prices, for calls will be determined first. Then, we derive lower bounds, or minimum prices, given other factors such as the stock price, strike price, and dividends. You will learn that the bounds for American calls frequently differ from those for European calls. After establishing the upper and lower pricing boundaries, we will prove that two calls that differ only in their times to expiration or only in their exercise price must satisfy certain pricing restrictions. In Section 16.3 we determine the analogous pricing restrictions for puts.

Along the way, important statements about the possibility of early exercise of American calls and puts will be made, given the pricing restrictions that were proven.

16.2 Pricing Restrictions for Calls

16.2.1 Upper Bound

The highest no-arbitrage price for a call is the current value of the underlying asset. That is,

Proposition I: $C \le S$

What if $C > S$

If Proposition I is violated, $C - S > 0$. The form of this inequality is $A > 0$. To arbitrage, we wish to trade today so that we receive a cash inflow of C and pay a cash outflow of S. Therefore, we sell the call and purchase the stock.

Today		**At Any Date Before Expiration, the Written Call May Be Assigned**[3]	**At Expiration (if not exercised early)** $S_T > K$	$S_T < K$
Sell call	$+C$	Deliver stock that you own	$-(S_T - K)$	0
Buy stock	$-S$	and receive K	$+S_T$	$+S_T$
	>0	$+K>0$	$+K$	$+S_T$

Thus, if a call price exceeds the stock price, an investor can arbitrage by writing a covered call. By purchasing the stock at a price that is lower than the written call premium, the arbitrageur is causing a cash inflow to occur at time zero. Subsequently, only positive or zero cash flows will occur. This is arbitrage.

It is important to account for the possible early assignment of a written American option. Thus, we include the center column of our arbitrage tables. If the written call had been European, only the initial and expiration day cash flows would have been relevant.

16.2.2 Lower Bounds

The lower bound for call prices depends on whether the call is American or European and on whether there will be ex-dividend days between today and the expiration date.

16.2.2.1 American and European Calls, No Dividends

Proposition II: $C \ge \max[0, S - K(1+r)^{-T}]$

Proposition II says that any call on a stock that pays no dividends (or will not pay any dividends until after the call has expired) must sell for more than zero if $S < K(1+r)^{-T}$. If $S > K(1+r)^{-T}$, the call must sell for more than the stock price minus the present value of K, $S - K(1+r)^{-T}$.

It should be intuitively obvious why option premiums must be nonnegative. If an option has a negative price, the buyer receives a cash inflow when he purchases the option. Obviously, if that situation ever presented itself, an investor could "buy" the option and throw it away. By doing so, he would realize a cash inflow. Therefore, we have proven that a call premium cannot be negative.

Now, we complete the proof by showing that a call must sell for more than the difference between S and the present value of K, that is, $C \geq S - K(1+r)^{-T}$.

What if	$C < S - K(1+r)^{-T}$
Then	$C - S + K(1+r)^{-T} < 0$
Or	$-C + S - K(1+r)^{-T} > 0$

		At Expiration	
Today		$S_T > K$	$S_T < K$
Buy call	$-C$	$+(S_T - K)$	0
Sell stock	$+S$	$-S_T$	$-S_T$
Lend	$-K(1+r)^{-T}$	$+K$	$+K$
	>0	0	$-S_T + K > 0$

This proof holds for American or European calls because the arbitrage involves the purchase of the call. Thus, because he owns the option, the arbitrageur does not have to worry about being assigned the exercise.

There is a weaker lower bound for American calls (not European calls) that would allow an arbitrageur to immediately realize the profit.

Proposition II′:	$C \geq \max[0, S - K]$
What if	$C < S - K$
Then	$C - S + K < 0$
Or	$-C + S - K > 0$

Today	
Buy call	$-C$
Exercise call (acquire the stock for $K)	$-K$
Sell stock that you bought from the exercise	$+S$
	>0

Proposition II′ states that an American call must always sell for at least its intrinsic value. This is a weaker bound than Proposition II, for in-the-money calls. To illustrate why it is weaker, consider the case of: $S=44$, $K=40$, $r=10\%$, and $T=1$ year. Proposition II says that for American or European calls on non-dividend paying stocks, $C \geq S - K(1+r)^{-T}$, that is, $C \geq 44 - 40/1.1 = 7.636$. Proposition II′, however, merely states that $C \geq S - K$, or $C \geq 44 - 40 = 4$.

Proposition II′ is a weaker bound than Proposition II because stating that the call premium must exceed 7.636 is stronger than saying that it exceeds a price of only 4. However, the proof of proposition II′ is interesting because all cash flows from the arbitrage are realized immediately. Also, Proposition II′ holds for all American calls, even if there will be ex-dividend days prior to expiration.[4]

We have now established upper and lower pricing bounds for American and European calls on non-dividend-paying stocks. Figure 16.2 illustrates the nature of these bounds. The upper

Figure 16.2 Boundaries on call option values before expiration, non-dividend paying stocks.

bound is the line $C=S$. The lower bound is the greater of 0 (if the call is out of the money), or $S-K(1+r)^{-T}$. The latter line dominates the $C=S-K$ line.

Before expiration, calls on non-dividend-paying stocks can sell only in the shaded region in Figure16.2 We can now make some important statements.

1. *Before expiration, in-the-money calls on non-dividend-paying stocks will always have some time value.* We know this because $C \geq S-K(1+r)^{-T}$. In Figure 16.2, note that in-the-money calls will always sell for more than their intrinsic value (if there are no dividends). On day T, there is no time to expiration, so $C_T=S_T-K$ if the call is in the money, and $C_T=0$ if the call is out of the money.

2. *An American call on a non-dividend-paying stock will never be exercised before expiration.* If an investor exercises an American call before expiration, an investor pays \$$K$ for stock worth \$$S$. That is, the "profit" is the call's intrinsic value, $S-K$. We know, however, that if the investor sells the call, he receives at least $S-K(1+r)^{-T}$. This latter amount is always greater than the call's intrinsic value. If the call is out of the money (i.e., $K<S$), it is irrational for an investor to exercise it early because the investor pays \$$K$ for the stock when exercising, but the market price is \$$S$. The only time an American call on a non-dividend-paying stock will be rationally exercised is on its expiration day.

3. *An American call on a non-dividend paying stock will sell for the same price as a European call on the same stock, all else equal.* If an American call will never be exercised early (under the assumptions made at the start of the chapter), then an American call is effectively the same as a European call. There is no value to the option to exercise early if it will never be used. Note that in the "real world," in-the-money American calls might be exercised early before expiration, even if the stock will not pay dividends prior to expiration. This is because of the market imperfections we assumed away at the start of the chapter. Example 16.1 illustrates how the bid-ask spread can lead to early exercise.

Taxes and commissions might also induce an investor to exercise early.[5] Finally, suppose an investor is short the stock and long an in-the-money call. To close the short position and eliminate the costly margin tied up in the short position, an investor might elect to exercise early.

EXAMPLE 16.1 The existence of bid–ask spreads might lead an investor to exercise early. Suppose $S=72$, $K=60$, and the bid and ask prices on the call are $11\,^3/_8$ and $12\,^1/_8$, respectively. If a market order to sell were placed, an investor would realize only $11\,^3/_8$. The investor might be better off exercising and selling the stock, even after commissions.

16.2.2.2 European Calls on Dividend-Paying Stocks

For simplicity, assume that there is just one ex-dividend day between today and the expiration day. Today is time 0, and the ex-day is time $t1$.[6] Under this assumption, we can state:

Proposition III:	$C \geq \max[0, S - \overline{D}(1+r)^{-t1} - K(1+r)^{-T}]$
What if	$C < S - \overline{D}(1+r)^{-t1} - K(1+r)^{-T}$
Then	$C - S + \overline{D}(1+r)^{-t1} + K(1+r)^{-T} < 0$
Or	$-C + S - \overline{D}(1+r)^{-t1} - K(1+r)^{-T} > 0$

			At Expiration	
Today		**At Time $t1$**	$S_T > K$	$S_T < K$
Buy call	$-C$		$+(S_T - K)$	0
Sell stock	$+S$	$-D$ (actual)	$-S_T$	$-S_T$
Lend	$-\overline{D}(1+r)^{-t1}$	$+\overline{D}$		
Lend	$-K(1+r)^{-T}$		$+K$	$+K$
	>0	≥ 0	0	>0

Note that there are two loans being made. First, a loan of $\$\overline{D}(1+r)^{-t1}$ is made from time 0 to time $t1$. This guarantees that the investor will receive the maximum dividend believed possible on the ex-day. The second loan is a loan of $\$K(1+r)^{-T}$ from time 0 to time T. Thus, the investor receives $\$K$ at time T.

At time $t1$, the stock trades ex-dividend. The arbitrageur is short the stock, so he must pay out the dividend. However, he had earlier made a loan so that he would be repaid an amount equal to the greatest dividend for which he believed he would be liable. Thus, at time $t1$, there is a positive cash flow (if the actual dividend is less than $\overline{D}$) or a zero cash flow (if the actual dividend equals $\overline{D}$).

EXAMPLE 16.2 As an example of a European call on a dividend paying stock that can sell below its intrinsic value, consider:

$S = 60$

$K = 20$

$T = 1$ year

$r = 10\%$

$K(1+r)^{-T} = 20/1.1 = 18.18$

Present value of dividends $= 3$

Note that while the intrinsic value of the call is 40, the lower bound of this call is only $60 - 18.18 - 3 = 38.82$. The reason for this is that the European call owner has a call only on the stock's value at time T, which equals the present value of all dividends subsequent to time T. The European call owner has no claim to any dividends paid between times 0 and T.

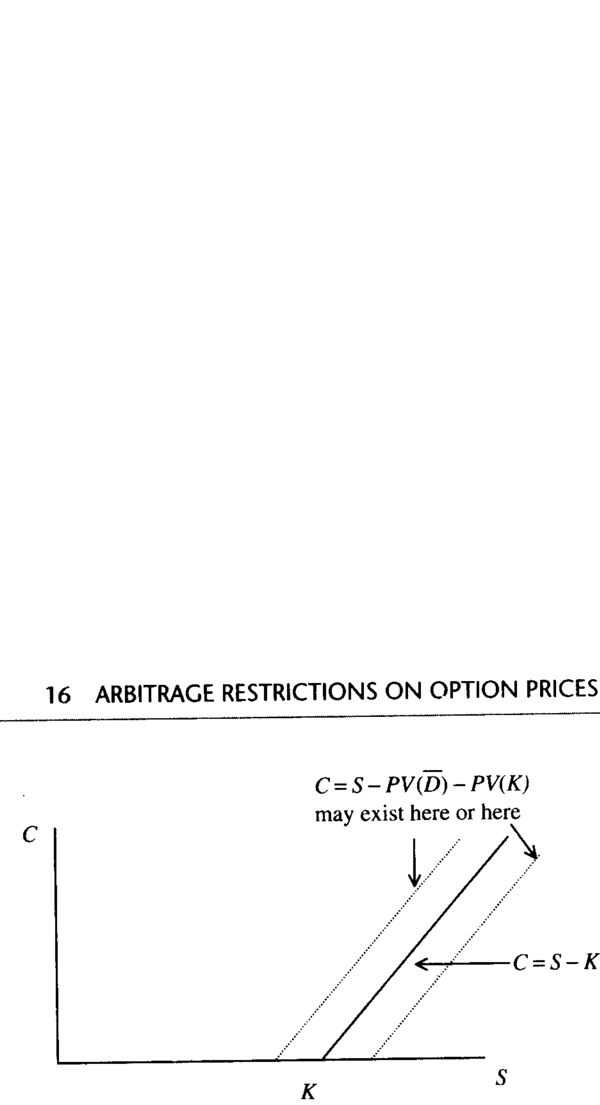

Figure 16.3 The lower bound for a European call on a dividend-paying stock can be to the left or the right of the intrinsic value line.

At expiration, the call must be sold for its intrinsic value if it finishes in the money, or else it expires worthless. The stock originally sold short must be covered (i.e., repurchased), creating a cash outflow of $\$S_T$. Finally, the original loan of $K(1+r)^{-T}$ is repaid, resulting in a cash inflow of $\$K$.

Proposition III creates an interesting phenomenon for European calls on dividend-paying assets. They can sell for below their intrinsic value. In other words, the restriction can lie to the left or the right of the $C=S-K$ line, as shown in Figure 16.3. Example 16.2 illustrates this phenomenon.

American calls, however, can never sell below their intrinsic value, regardless of dividends. If a dividend before the expiration date of the call is significant, the American call owner will exercise and capture the dividend.

16.2.2.3 American Calls on Dividend-Paying Stocks

As stated in Proposition IV, to prevent arbitrage the value of an American call on a dividend paying stock must equal or exceed four values. That is, the lower bound on the value of such a call is given by the maximum of the four values. In Proposition IV, we assume that times $t1$ and $t2$ are the nearby and more distant ex-dividend dates, respectively. If there is only one ex-date, or more than two ex-dates, Proposition IV must be appropriately modified (see page 451–2 for an example using three ex-dates).

Propositions IV: $$C \geq \max \begin{cases} \text{a.} & 0 \\ \text{b.} & S-K(1+r)^{-t1} \\ \text{c.} & S-\overline{D1}(1+r)^{-t1}-K(1+r)^{-t2} \\ \text{d.} & S-\overline{D1}(1+r)^{-t1}-\overline{D2}\,(1+r)^{-t2}-K(1+r)^{-T} \end{cases}$$

We have already demonstrated that a call cannot sell for a negative value. Thus, Proposition IVa has been proven.

Proof of Proposition IVb proceeds as follows.

What if	$C<S-K(1+r)^{-t1}$
Then	$C-S+K\,(1+r)^{-t1}<0$
Or	$-C+S-K(1+r)^{-t1}>0$

Today		Just Before the Stock Trades Ex-Dividend at Time $t1$
Buy call	$-C$	Exercise call: acquire stock and pay K
Sell stock	$+S$	Long stock and short stock offset each other
Lend	$-K(1+r)^{-t1}$	$+K$
	>0	0

The arbitrageur exercises the call an instant before the stock trades ex-dividend. In reality, if the stock traded ex-dividend on, say, January 12, then the call would be exercised at the close of trading on January 11. By exercising, the arbitrageur pays $K for the long stock position. However, because the stock was also sold short, the two positions offset each other.

Proof of Proposition IVc involves the exercise of the call just before the stock trades ex-dividend at time $t2$. It is left as an exercise for the student. However, the proof of Proposition IVd should provide a guide to prove Proposition IVc.

Proposition IVd proceeds as follows.

What if $C < S - \overline{D1}(1+r)^{-t1} - \overline{D2}(1+r)^{-t2} - K(1+r)^{-T}$

Then $C - S + \overline{D1}(1+r)^{-t1} + \overline{D2}(1+r)^{-t2} + K(1+r)^{-T} < 0$

Or $-C + S - \overline{D1}(1+r)^{-t1} - \overline{D2}(1+r)^{-t2} - K(1+r)^{-T} > 0$

Today		Time $t1$	Time $t2$	Time T: $S_T>K$	Time T: $S_T<K$
Buy call	$-C$			$+(S_T-K)$	0
Sell stock	$+S$	$-D1$(actual)	$-D2$(actual)	$-S_T$	$-S_T$
Lend	$-\overline{D1}(1+r)^{-t1}$	$+\overline{D1}$			
Lend	$-\overline{D2}(1+r)^{-t2}$		$+\overline{D2}$		
Lend	$-K(1+r)^{-T}$			$+K$	$+K$
	>0	≥ 0	≥ 0	0	>0

Let's work an example of Proposition IVd. On March 20, 1997, IBM stock closed at $146 $^1/_8$ per share. The January 1998 IBM LEAP call with $K=130$ closed at 27. The last dividend amount was $0.35/share, and the last ex-dividend date was February 6, 1997. The option expires on the third Friday of January 1998, which is January 16. The riskless interest rate is 6%. Assume that the earliest next three ex-dates will be on May 6, August 6, and November 6, 1997, and that the highest dividend amounts possible are $0.40, $0.45, and $0.45 per share on those three dates, respectively. Thus,

$S = 146\ ^1/_8$

$K = 130$

$T = 302$ days $= 0.8274$ year

$r = 6\%$/year

$t1 = 47$ days $= 0.1288$ year
$t2 = 139$ days $= 0.3808$ year
$t3 = 231$ days $= 0.6329$ year
$\overline{D1} = 0.40$

$\overline{D2} = 0.45$

$\overline{D3} = 0.45$

Because there are three dividend dates, Propositions IV state that the call must sell for more than the greatest value of the following *five* amounts (note that to account for three dividend dates, we must modify Proposition IV on page 450).

a. 0

b. $S - K(1+r)^{-t1} = 146.125 - 130/(1.06)^{0.1288} = 17.097$

c. $S - \overline{D1}(1+r)^{-t1} - K(1+r)^{-t2} = 146.125 - 0.40/(1.06)^{0.1288} - 130/(1.06)^{0.3808} = 18.5808$

d. $S - \overline{D1}(1+r)^{-t1} - \overline{D2}(1+r)^{-t2} - K(1+r)^{-t3}$
$= 146.125 - 0.40/(1.06)^{0.1288} - 0.45/(1.06)^{0.3808} - 130/(1.06)^{0.6329} = 19.9947$

e. $S - \overline{D1}(1+r)^{-t1} - \overline{D2}(1+r)^{-t2} - \overline{D3}(1+r)^{-t3} - K(1+r)^{-T}$
$= 146.125 - 0.40/(1.06)^{0.1288} - 0.45/(1.06)^{0.3808} - 0.45/(1.06)^{0.6329}$
$- 130/(1.06)^{0.8274} = 20.973$

The greatest value of these four amounts is 20.973, and the actual call value was 27. Thus, there were no arbitrage opportunities. However, suppose the actual call value was 20. Then the price would violate Proposition IVe. To arbitrage, do the following:

					Jan. 16, 1997	
March 20		**May 6**	**Aug. 6**	**Nov. 6**	**$S_T > 130$**	**$S_T < 130$**
Buy call	-20				$+(S_T - 130)$	0
Sell stock	$+146.125$	$-D1$(actual)	$-D2$(actual)	$-D3$(actual)	$-S_T$	$-S_T$
Lend	-0.3970	$+0.40$				
Lend	-0.4401		$+0.45$			
Lend	-0.4337			$+0.45$		
Lend	-123.88				$+130$	$+130$
	$+0.9742$	≥ 0	≥ 0	≥ 0	0	≥ 0

Note that the arbitrage does not require early exercise of the call. There will be no cash outflows at any date. There is a cash inflow of \$0.9742 on March 20. If the actual dividend on May 6 is less than 0.40, there will be another cash inflow on that date because \$0.40/share was the highest dividend the arbitrageur thought possible on May 6. If the stock price on the expiration date is greater than 130, the arbitrageur gets a zero cash flow on that day. It is also possible that if $S_T < 130$, an additional cash inflow will be earned at expiration.

Now we can make some important statements

1. *An in-the-money American call on a dividend-paying stock will always have some time value, except (possibly) on the day before it trades ex-dividend and (always) on its expiration day.* This statement can be made because of Proposition IVb. An American call that is in the money will always sell for at least $S - K(1+r)^{-t1}$, and this amount always exceeds $S - K$ (its intrinsic value) except on the last with-dividend day, when $t1$ approaches 0. If Proposition IVb is the lower bound of an American call on a dividend-paying stock, then it might have no time value on the day before it trades ex-dividend.

2. *An in-the-money American call will never be exercised early, except on the day before it trades ex-dividend.* By exercising an American call early, the call owner realizes only the call's intrinsic value. Before the stock's last "with-dividend" day, it is preferred to sell the call under our assumptions. This is because the intrinsic value *and* some time value will be realized. However, on the day before the stock trades ex-dividend, it may have only intrinsic value. This is a necessary condition for it to be exercised early. Again, it must be stressed that market imperfections such as transactions costs and taxes might result in some investors exercising in the money calls earlier than the day before it trades ex-dividend.

16.2.3 Early Exercise of American Calls[7]

Before expiration, the only possible time that an American call will (theoretically) have zero time value is the last day before an ex-dividend date. Therefore, we can conclude that American options will never be exercised except immediately before any ex-dividend dates, and at expiration. Recall that when the call owner sells the option, he receives its intrinsic value plus some time value. If the owner exercises the option, however, he receives only its intrinsic value.

Someone who exercises too early forfeits the option's time value and also, if he continues to retain ownership in the stock may regret having exercised. The regret comes from a sudden and unexpected large decline in the stock price. That is, if the stock price is below $K at expiration, the owner of the call has realized the benefits of a call's limited liability because he loses the same amount if S_T is a dollar below $K or if S_T drops to zero. A stock owner, however, bears the entire loss.

Finally, funds must be used to pay $K for the stock, and these funds will no longer be able to earn a rate of return for the early exerciser. One condition for optimal early exercise is for the dividend amount to exceed the present value of the interest that could be earned on $K between the ex-date and the option's expiration date. An investor who is considering early exercise on the last with-dividend date will receive the dividend but will lose the interest he could have earned on the $K that he is effectively saving in the bank until expiration.

Thus, we have stated three reasons for not exercising until just before the stock trades ex-dividend (in practice, just before the close of trading on the last with-dividend day). The benefit of exercising on this day is that the call owner will receive the dividend. Exercising early on any day other than the day before the ex-day has the three costs just stated, but no benefits. For American calls, the early exercise privilege is valuable only on the last with-dividend date. On that date, the call owner captures the dividend by exercising early. We next show that by exercising early when it is optimal, he will also avoid a decline in the value of the call on the ex-dividend day.

Assume the following: (a) an investor owns an in-the-money American call on a stock that is about to trade ex-dividend; (b) the stock will fall in price by the amount of the ex-dividend amount; and (c) the call is selling at its minimum price boundary condition defined by Proposition IVb (i.e., it has no time value). On the ex-day, the call may trade at a lower price because the stock price will fall, and because the lower pricing boundary has fallen from

$$C \geq S(\text{with-div}) - K$$

an instant before it trades ex-dividend to

$$C \geq S(\text{ex-div}) - K(1+r)^{-T}$$

where T is the time to expiration and $S(\text{ex-div}) = S(\text{with-div}) - \text{div}$.

In other words, the lower pricing boundary actually shifts on the ex-dividend day. The possibility of arbitrage keeps the American call value high (just equal to, or just above, its intrinsic value) on the last with-dividend day, when an otherwise equivalent European call would have sold at less than its intrinsic value because of the impending dividend. After the stock trades ex-dividend, and if there are no more ex-dates, the European and American calls will sell for the same price, and both must sell for more than intrinsic value.

Figure 16.4 shows how an American call can fall in value on the ex-dividend day. On the last with-dividend day, the stock price is S(with-div), and the call is selling at its lowest possible price, $C(\text{with-div}) = S(\text{with-div}) - K$. This is the lower bound by Proposition IVb, when $t1$ is zero. Figure 16.4 shows the stock price falling on the ex-dividend day to S(ex-div). If there are no further dividends, the new lower boundary for the call is $S - K(1+r)^{-T}$ (by Proposition II). Figure 16.4 shows a range of possible ex-day call values that are below C(with-div). No investor who expected the ex-dividend call price to be in this range would hold the call. If all investors shared the same belief that the call will decline in value to be in this range, no other investors would be willing to buy options from the call owners. Thus, early exercise is the only way to avoid the loss.

Let's make a general demonstration of the early exercise logic. A call owner will exercise her call early if the realized wealth from exercising exceeds the value of the call if it is held another day (on which the underlying asset will trade ex-dividend). If the call is exercised early, her wealth will be $S(\text{with-div}) - K$. Thus, the call owner will exercise early on the day before the stock trades ex-dividend if the call's intrinsic value on that day is greater than the call price expected on the ex-date:

$$S(\text{with-div}) - K > E[\tilde{C}(\text{ex-dividend})] \tag{16.1}$$

The tilde (˜) above the ex-dividend call value denotes it as a random variable. Note also that on the day before a stock trades ex-dividend, deep in-the-money calls will often sell for their lower bound of $S(\text{with-div}) - K$.

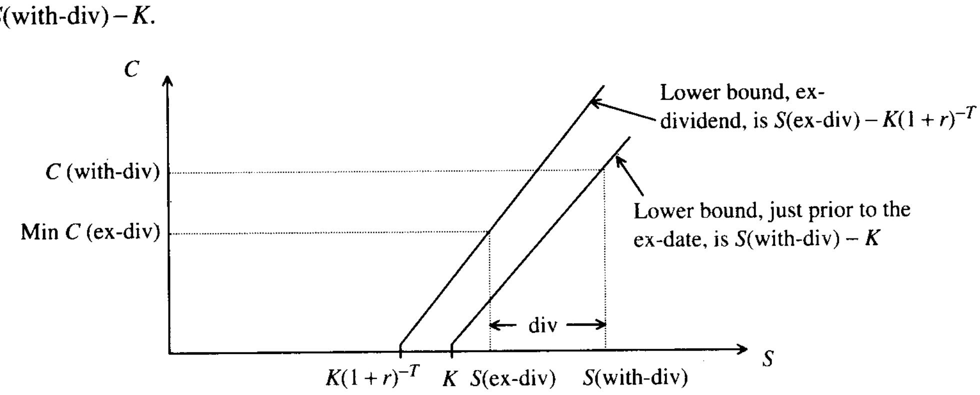

Figure 16.4 American calls on dividend-paying stocks should be exercised early (on the last with-dividend day) when the value of the call will decline on the ex-dividend day.

The ex-dividend call value will sell for its lower bound, perhaps plus an unknown amount, $\tilde{z}$:

$$E[\tilde{C}(\text{ex-div})] = E[\tilde{S}(\text{ex-div}) - K(1+r)^{-T} + \tilde{z}] \qquad (16.2)$$

Substitute Equation (16.2) into (16.1). We conclude that the call owner will surely exercise if:

$$S(\text{with-div}) - K > E[\tilde{S}(\text{ex-div}) - K(1+r)^{-T} + \tilde{z}] \qquad (16.3)$$

The variable $\tilde{z}$ allows the call to sell for more than its minimum bound on the ex-dividend day. Now assume that the stock price is expected to decline by the dividend amount on the ex-dividend day and write

$$E[\tilde{S}(\text{ex-div})] = S(\text{with-div}) - \text{div} \qquad (16.4)$$

Substituting Equation (16.4) into (16.3), we conclude that early exercise is optimal if

$$S(\text{with-div}) - K > S(\text{with-div}) - \text{div} - K(1+r)^{-T} + E[\tilde{z}] \qquad (16.5)$$

After simplifying this expression, we have:

$$\text{div} > K - K(1+r)^{-T} + E[\tilde{z}] \qquad (16.6)$$

This expression must hold for early exercise to be optimal. Note that if the call is sufficiently deep in the money and/or close to expiration, it is likely that the ex-dividend call will sell for its lowest possible value of $S - K(1+r)^{-T}$, which means that for deep in-the-money, short-term calls, $E[\tilde{z}]$ often will be zero. If $E[\tilde{z}] = 0$, then the condition for optimal early exercise is:

$$\text{div} > K - K(1+r)^{-T} \qquad (16.7)$$

The right-hand side of Equation (16.7) equals the interest that could be earned by investing the present value of $$K$ until expiration. Thus, we have shown that a deep in-the-money call should be exercised just before it trades ex-dividend if it is expected to decline in value when the stock trades ex dividend, and this will occur when condition (16.7) exists. Equivalently, we can say that early exercise is optimal when the proceeds from early exercise exceed the expected ex-day price of the call.

EXAMPLE 16.3 If $K = 50$, $r = 10\%$/year, and $T = 2$ months, then $K - [K/(1+r)^T] = 50 - [50/(1.1)^{0.1667}] = 0.7880$, which is the present value of the interest on \$50. If (a) the dividend exceeds this amount, (b) the call owner believes that the stock price will fall by the dividend amount, and (c) the call owner believes that the call will sell at its lower bound on the ex-day, the call should be exercised.

Consider the data more specifically. Let the with-dividend stock price be \$60 and the dividend equal \$2. If the American call is selling at its lower bound, with-dividend, it sells for \$10. If you expect the call, ex-dividend, to sell at its lower bound of $S(\text{ex-div}) - K(1+r)^{-T}$, then you expect it to be worth only $58 - 49.212 = 8.788$. In other words, holding the call will produce a loss. If all investors have the same beliefs as you, no one will buy the call from you, with-dividend, for \$10. The only rational action for all owners of this call is to exercise the calls, with-dividend.

Summing up, early exercise of American calls will occur only on the day before an ex-dividend date and is more likely to occur

The greater the dividend
The lower the interest rate
The more the call is in the money
The lower the strike price
The shorter the remaining time to expiration
The lower the time value of the call on the last with-dividend day

16.3 Puts

16.3.1 Upper Bounds

16.3.1.1 American Puts

Proposition V:	$P \leq K$
What if	$P > K$
Then	$P - K > 0$

Today		**At Any Later Date if the Written Put Is Assigned**
Sell put	$+P$	Pay K to acquire the stock, $-K$ Immediately sell the stock, $+S$
Lend	$-K$	$+K+$ interest
	>0	$+S+$ interest >0

Even if S is zero, the arbitrageur realizes a cash inflow equal to the interest on the loan of $\$K$. If the written put is never exercised, the arbitrageur receives a positive cash flow at time T of $\$K+$ interest.

16.3.1.2 Upper Bound for European Puts

For European puts, we can obtain an even tighter upper boundary for prices:

Proposition VI:	$P \leq K(1+r)^{-T}$
What if	$P > K(1+r)^{-T}$
Then	$P - K(1+r)^{-T} > 0$

		At Expiration	
Today		$S_T > K$	$S_T < K$
Sell put	$+P$	0	$-(K - S_T)$
Lend	$-K(1+r)^{-T}$	$+K$	$+K$
	>0	$+K>0$	$+S_T>0$

EXAMPLE 16.4 Consider the following data:

$K = 20$

$K(1+r)^{-T} = 19$

American put premium = 19.50

This situation does *not* offer an arbitrage opportunity. Suppose an investor tried to arbitrage by selling the put (cash inflow of 19.50) and lending (cash outflow of 19). Then, the next day, the stock falls to zero and the put is exercised early. The investor receives worthless stock and pays K (cash outflow of 20). However, the loan of 19 will have earned only one day's interest. The cash flow on the early exercise date is then negative (19 + one day's interest + 0.50 − 20). Thus, a violation of Proposition VI does not lead to arbitrage profits for American puts.

It is important to note that Proposition VI holds for European puts. If an investor tried to arbitrage with an American put when $K(1+r)^{-T} < P < K$, the written American put could be assigned before expiration at a time that the stock price was very low (consider the case if the stock were worthless). In this case, interest earned on the loan of $K(1+r)^{-T}$ might not be sufficient to fund the required payment, which is equal to the strike price. As a result, this could lead to a negative cash flow on the early exercise date.

16.3.2 Lower Bound for Puts on a Non-Dividend-Paying Stock

16.3.2.1 American Puts

Proposition VII: $P \geq \max[0, K - S]$

Like calls, puts cannot be worth a negative amount. If a put option "sold" for less than zero, an investor could buy the put, receive a cash inflow equal to the negative premium, and then throw the put away.

What if	$P < K - S$?
Then:	$P - K + S < 0$
Or:	$-P + K - S > 0$

Today

Buy put	$-P$
Buy stock	$-S$
Exercise put	$+K$
	>0

Note that this arbitrage can be done only with American puts because the purchased put is immediately exercised. Moreover, the transaction could be done at any time, regardless of dividend payments. Thus, the following statement can be made:

An American put can never sell below its intrinsic value. It will always have zero or positive time value.

16.3.2.2 European Puts

For European puts on a non-dividend-paying stock, we have a weaker lower pricing boundary:

Proposition VIII:	$P \geq \max\,[0, K(1+r)^{-T} - S]$
What if	$P < K(1+r)^{-T} - S$
Then	$P - K(1+r)^{-T} + S < 0$
Or	$-P + K(1+r)^{-T} - S > 0$

		At Expiration	
Today		$S_T > K$	$S_T < K$
Buy put	$-P$	0	$+(K - S_T)$
Borrow	$+K(1+r)^{-T}$	$-K$	$-K$
Buy stock	$-S$	$+S_T$	$+S_T$
	>0	≥ 0	0

Thus, from Proposition VIII:

A European put on a non-dividend-paying stock can sell for less than its intrinsic value.

In fact, in-the-money European puts will frequently sell for less than their intrinsic value. These puts will also rise in value as time passes, since we know that at expiration, they must sell for their intrinsic value, $\max[0, K - S_T]$. American puts can never sell for less than their intrinsic value and can never rise in value as the expiration date nears.

Summarizing what we have learned for puts on non-dividend-paying stocks, there are different boundaries for American puts and for European puts. They are depicted in Figure 16.5.

Note that American puts have a higher price range than European puts. This is intuitive when the value of the right to exercise early is high. Because in-the-money American puts are, in fact, frequently exercised early, they will often sell for more than otherwise equivalent European puts.

At least two factors induce an owner of an in-the-money American put to exercise early. If the put is deeply in the money, its owner can begin to earn a return on the $K he receives from exercising the put. Also, as S declines, the range of "good" outcomes for the put owner narrows. After all, the stock can only go to zero. For example, why would one want to buy a put with a $10 strike

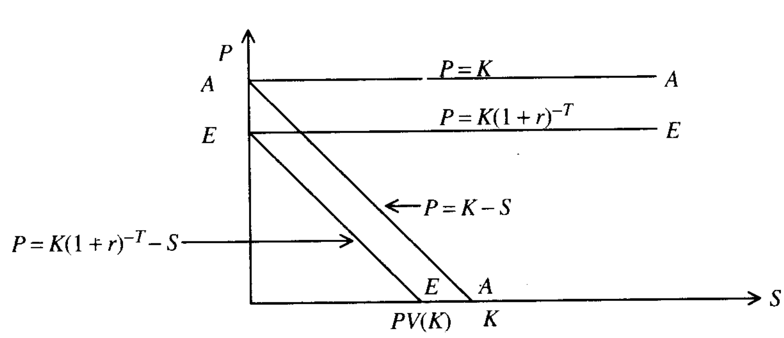

Figure 16.5 Pricing boundaries for puts on non-dividend-paying stocks: *AAA*, American put boundaries; *EEE*, European put boundaries.

price if the stock were selling for only \$0.50 per share? There is very limited additional profit to be earned if the stock price fell to zero.

16.3.3 Lower Price Boundaries for Puts on Dividend-Paying Stocks

16.3.3.1 European Puts

$$\textbf{Proposition IX:}\quad P \geq \max[0, K(1+r)^{-T} - S + \underline{D}\,(1+r)^{-t1}]$$

Recall that $\underline{D}$ is the smallest dividend believed possible. If there is more than one dividend, the present value all the smallest dividends believed possible must be added to Proposition IX.[8]

What if	$P < K(1+r)^{-T} - S + \underline{D}(1+r)^{-t1}$
Then	$P - K(1+r)^{-T} + S - \underline{D}(1+r)^{-t1} < 0$
Or	$-P + K(1+r)^{-T} - S + \underline{D}(1+r)^{-t1} > 0$

			At Expiration	
Today		**At Time $t1$**	$S_T > K$	$S_T < K$
Buy put	$-P$		0	$+(K - S_T)$
Borrow	$+K(1+r)^{-T}$		$-K$	$-K$
Buy stock	$-S$	$+\underline{D}$(actual)	$+S_T$	$+S_T$
Borrow	$+\underline{D}(1+r)^{-t1}$	$-\underline{D}$		
	>0	≥ 0	≥ 0	0

Dividends increase the lower bound of European put prices. The minimum price is raised by the present value of the smallest dividend that the arbitrageur believes will be paid ($\underline{D}$ equals zero if there is a probability that the dividend will be skipped altogether). This result is intuitive, since on a stock's ex-dividend day, the price falls by the amount of the dividend. The price decline will make the put more valuable. The boundary of Proposition VIII can never be weaker than the boundary of Proposition IX.

16.3.3.2 American Puts

The lower bound for American puts on dividend-paying stocks is the greatest of several possible amounts. Let us assume that there are two ex-dividend dates before the expiration date of the put. Then, the put must be worth the maximum of four values:

$$\textbf{Proposition X:}\quad P \geq \max\begin{cases} \text{a.}\ \ 0 \\ \text{b.}\ \ K - S \\ \text{c.}\ \ K(1+r)^{-t1} - S + \underline{D1}\,(1+r)^{-t1} \\ \text{d.}\ \ K(1+r)^{-t2} - S + \underline{D1}(1+r)^{-t1} + \underline{D2}\,(1+r)^{-t2} \end{cases}$$

We already know that an American put must sell for more than zero and for more than $K-S$ (Proposition VII). In proving parts Propositions Xc and Xd, the arbitrageur would exercise the put an instant *after* the stock trades ex-dividend. To facilitate understanding of the proof, we will use the notation $t1+\varepsilon$ to denote "an instant after the ex-dividend time." In practice, it refers to the opening of trading on the ex-day. There is no difference, however, between the present value of \$1 received at time $t1$, and \$1 received at time $t1+\varepsilon$ because ε is "an instant."

Proof of proposition Xc proceeds as follows.

What if $P<K(1+r)^{-t1}-S+\underline{D1}(1+r)^{-t1}$

Then $P-K(1+r)^{-t1}+S-\underline{D1}(1+r)^{-t1}<0$

Or $-P+K(1+r)^{-t1}-S+\underline{D1}(1+r)^{-t1}>0$

Today		At Time $t1$	At Time $t1+\varepsilon$ (exercise the put and deliver the shares owned)
Buy put	$-P$		$+K$
Borrow	$+K(1+r)^{-t1}$		$-K$
Buy stock	$-S$	$+D1$(actual)	
Borrow	$+\underline{D1}(1+r)^{-t1}$	$-\underline{D1}$	
	>0	≥ 0	0

If the put has any time value at time $t1+\varepsilon$, it will be better to sell the put, sell the stock, and repay the loan principal of K (verify this for two cases: if the put at time $t1+\varepsilon$ is in the money and if it is out of the money). This would still generate additional cash inflows.

The proof of Proposition Xd is essentially identical to that of Xc and is left as an exercise. The arbitrageur would exercise the put at time $t2+\varepsilon$, an instant after the second ex-dividend date.

Any of the three lower bounds Proposition (Xb, Xc, or Xd) may apply for an in-the-money put. Consider the following three-part example:

$S=20$

$K=30$

$t1$ = first ex-dividend date = 1 month = 0.08333 year

$t2$ = second ex-dividend date = 4 months = 0.3333 year

r = 10%/year

Case I: $\underline{D1}=\underline{D2}$ = smallest dividend amount at times $t1$ and $t2=1.0$

Case II: $\underline{D1}=\underline{D2}$ = smallest dividend amount at times $t1$ and $t2=0.3$

Case III: $\underline{D1}=\underline{D2}$ = smallest dividend amount at times $t1$ and $t2=0.1$

Given that P must exceed zero and must be equal to or exceed the maximum of the values computed in Xb, Xc, and Xd above, we have:

For Case I:

$$P \geq \max[0, 30-20, 30(1.1)^{-0.0833}-20+1(1.1)^{-0.0833},$$
$$30(1.1)^{-0.3333}-20+1(1.1)^{-0.0833}+1(1.1)^{-0.3333}] \quad \text{(Xd is boundary)}$$
$$= P \geq \max[0,\ 10,\ 10.7548,\ 11.0227] = P \geq 11.0227$$

For Case II:

$$P \geq \max[0, 30-20, 30(1.1)^{-0.0833}-20+0.3(1.1)^{-0.0833},$$
$$30(1.1)^{-0.3333}-20+0.3(1.1)^{-0.0833}+0.3(1.1)^{-0.3333}] \quad \text{(Xc is boundary)}$$
$$= P \geq \max[0, 10, 10.0603, 9.6501] = P \geq 10.0603$$

For Case III:

$$P \geq \max[0, 30-20, 30(1.1)^{-0.0833} - 20 + 0.1(1.1)^{-0.0833},$$
$$30(1.1)^{-0.3333} - 20 + 0.1(1.1)^{-0.0833} + 0.1(1.1)^{-0.3333}] \quad \text{(Xd is boundary)}$$
$$= P \geq \max[0, 10, 9.8619, 9.2580] = P \geq 10$$

Now, suppose that the conditions cited in Case I held, and a put expiring in five months sold for 10.50. An arbitrageur could buy the stock and buy the put, for an initial cash outlay of 30.50. He can borrow that amount for four months (because he plans on exercising immediately after receiving the second dividend), agreeing to repay $[30.5(1.1)^{0.333} =]$ \$31.485 at time $t2+\varepsilon$. Thus, there is no initial cash outlay required for the arbitrage. At time $t1$, he would receive the first dividend of \$1 and invest it to earn 10% for three months. On the option's expiration day, this will be worth $[1(1.1)^{0.25} =]$ \$1.024. At time $t2$, he will receive another dollar in dividends, and immediately thereafter, he will exercise the put and receive the strike price of 30. His outflow at time $t2+\varepsilon$ is the repayment of the initial loan principal, and interest of \$31.485. However, he will also have three sources of cash inflows at time $t2+\varepsilon$: the strike price from exercising the put (\$30), the first dividend and interest earned on it (\$1.024), and the second dividend (\$1), for a total cash inflow of \$32.024. Thus, there is an arbitrage opportunity. If the stock price were to rise above \$30/share by time $t2+\varepsilon$, even more profits would be earned. Likewise, if dividend payments are actually more than \$1/share at times $t1$ or $t2$, the arbitrage profits will increase.[9]

16.3.4 Early Exercise of American Puts

Frequently, it is optimal for owners of in-the-money American puts to exercise their options early. Consider the decision facing the owner of an in-the-money put who is also very bearish. He believes with certainty (i.e., probability = 1.0) that the expiration day stock price will be below the strike price. Suppose the put is selling for its intrinsic value, $K-S$. The investor has three possible courses of action: he can hold, sell, or exercise the option. Clearly, holding the put option for another day is an inferior strategy in comparison to exercising. By exercising, the investor receives \$$K$ today, and he can immediately invest it to earn interest. By waiting one day, or waiting until the expiration day, he is forgoing interest that could be earned on the \$$K$.

Now, all we have to do is add the assumption that all investors agree that there is no chance that the put will finish out of the money. In that case, no one would ever want to buy the put for $K-S$ and hold it. The put essentially becomes a "hot potato." In other words, an investor may be unable, at certain times, to sell a deep in-the-money American put for its intrinsic value.[10]

In Chapter 17 (Section 17.4), we will use the binomial option pricing model to demonstrate that an investor can frequently use stocks and bonds to replicate a deep in-the-money American put at a cost less than the price of the American put. In other words, an investor could arbitrage by exercising an American put (realizing its intrinsic value of $K-S$), and then proceeding to create an equivalent put for a cost less than $K-S$.

Summarizing, early exercise of American puts will be more likely:

- The higher the strike price
- The lower the stock price
- The higher the interest rates

- The smaller the time value
- The more deeply the put is in the money

Dividends will tend to reduce the likelihood of early exercise.[11] One reason is that dividends may increase the lower bound of American put prices (see Section 16.3.3.2). Thus, ex-dividend dates might create more time value for a put. Moreover, even the bearish owner of both a put and the underlying stock (a protective put) might decide to wait to receive the next dividend before exercising his put. If the present value of the next dividend exceeds the present value of the interest that could be earned on \$$K$ between today and the ex-date, an investor might decide not to exercise early but to wait for the dividend.

Even an American put with time value could be exercised early. Exercising early does destroy the put's time value. But the interest earned on \$$K$ (more accurately, the interest earned on the present value of \$$K$) could exceed the time value that is killed if it is exercised early. Thus, the following statement can be made:

An in-the-money American put can rationally be exercised at almost any time before expiration.

16.4 Put–Call Parity

Put–call parity establishes a no-arbitrage pricing condition between a call and a put premium. This pricing boundary of a call relative to a put on the same stock requires that both options have the same strike price and time to expiration. Put–call parity also has importance in obtaining an exact European put option pricing model in subsequent chapters and provides, as well, some insights into certain attributes of options. Although put–call parity has long been known to option traders, Stoll (1969) formalized its proof. In this section, all the assumptions previously made on the first page of this chapter still apply.

16.4.1 European Options; No Dividends

The most basic put–call parity proposition applies to European puts and calls on stocks that will pay no dividends before option expiration. In words, the difference between the price of a call and a price of a put on the same stock, with the same strike price and time to expiration, equals the price of the underlying asset minus the present value of the strike price. That is:

Proposition XI: $C-P=S-K(1+r)^{-T}$

We divide the proof into parts A and B.

Part A

What if $C-P>S-K(1+r)^{-T}$

Then $C-P-S+K(1+r)^{-T}>0$

		At Expiration	
Today		$S_T<K$	$S_T>K$
Sell call	$+C$	0	$-(S_T-K)$
Buy put	$-P$	$+(K-S_T)$	0
Buy stock	$-S$	$+S_T$	$+S_T$
Borrow	$+K(1+r)^{-T}$	$-K$	$-K$
	>0	0	0

Part A of the proof illustrates a **conversion** trade. In a conversion, an arbitrageur exploits the violation of the put–call parity relationship by buying the stock, selling the overpriced call, buying the underpriced put, and borrowing. However, quite often no money is borrowed. Instead, the arbitrageur realizes a riskless rate of return in excess of what can be earned on other riskless investments.

Part B

What if	$C-P<S-K(1+r)^{-T}$
Then	$C-P-S+K(1+r)^{-T}<0$
Or	$-C+P+S-K(1+r)^{-T}>0$

		At Expiration	
Today		$S_T<K$	$S_T>K$
Buy call	$-C$	0	$+(S_T-K)$
Sell put	$+P$	$-(K-S_T)$	0
Sell stock	$+S$	$-S_T$	$-S_T$
Lend	$-K(1+r)^{-T}$	$+K$	$+K$
	>0	0	0

Part B of the proof illustrates a **reverse conversion**, or **reversal** trade. In this part of the proof, we assume that the arbitrageur receives full use of the proceeds of the short sale. However, most individuals will not have this advantage. However, "quasi-arbitrageurs" already own the stock, and they have the ability to sell the stock and receive the proceeds.

There is some additional useful discussion of Proposition XI. First, arrange it to read:

$$-C=-S-P+K(1+r)^{-T}$$

$$\text{buy call} = \text{buy stock} + \text{buy put} + \text{borrow } K(1+r)^{-T}$$

EXAMPLE 16.5 Suppose an April 45 Dell Computer call sells for 4 7/8, the April 45 put is at 2 7/8, and Dell stock price is 47. Assume that the riskless interest rate is 6%/year and that there are 39 days to expiration (i.e., 0.10685 year). Do these option prices obey put–call parity? From Proposition XI we write:

$$C-P=S-K(1+r)^{-T}$$

$$4.875-2.875=47-45(1.06)^{-0.10685}$$

$$2<2.287$$

This is close, but not exact. The violation of Proposition XI signals that the call is "cheap" relative to the put. Thus, the arbitrage trades are to buy the call, write the put, sell the stock, and lend the proceeds. However, the following conditions are essential: (1) these must be European options, or (2) it must be known that Dell Computer will pay no dividends in the next 39 days. To conditions 1 or 2 we add two others: (3) we were sure that we could trade at these prices, and (4) we could cover all trading costs (whew!).

You may have heard that a call option has properties similar to a levered purchase of stock (i.e., buying stock on margin). For example, consider two alternatives: (a) buy a call on 100 shares for \$300, or (b) buy 100 shares of stock for \$30/share by borrowing \$2400 from your broker and investing \$600 of your own money. Now, what would happen if the very next day, the stock rose to \$31/share? The value of a typical at-the-money call might rise to \$350, which represents a 16.667% rate of return (note that the stock itself rose only 3.333%). Under alternative (b), the investor would repay the loan to the broker. Ignoring the one day of interest, the investor is left with (\$3100 − \$2400 =) \$700. Thus, the investor has also earned a return of 16.667% [(700 − 600)/600]. This example illustrates that a call purchase has similar characteristics to a levered purchase of the underlying asset.[12]

However, the call owner also effectively owns a put. The put gives the call owner limited liability. If the price of the underlying asset were to fall below the strike price, and finish out of the money, the call owner would not lose any additional capital beyond what he originally paid for the call. This is unlike the levered equity position, where the investor could lose more than the initial investment of \$600 if the stock price were to decline below \$24/share.

A second interesting result is obtained by studying put–call parity in the form $C-P=S-K(1+r)^{-T}$. For an interest rate $r=0$, this equation reduces to $C-P=S-K$. However, for positive interest rates, the value of an at-the-money call will always exceed the value of a put if the two European options are on the same underlying asset and have the same K and T, and there are no dividends to be paid prior to expiration.

A third implication of the basic put–call parity proposition is that floor traders and other exchange members (who have low transactions costs) can use the principles behind put–call parity to create their own puts. Before June 1977, there were no exchange-traded put options. Thus, put positions *had* to be used to create conversions and reversals. At times it may be cheaper to replicate an option via a conversion or reversal than by buying or selling the actual option.

Finally, there is an implication from put–call parity for the role of market expectations in determining option values. When investors are bullish, one would intuitively expect call values to rise. Similarly, you might expect put values to rise if investors are bearish. Under the perfect market conditions used to derive put–call parity, market expectations cannot influence option values. Even if investors were wildly bullish about the prospects for the underlying stock, $C-P$ must equal $S-K(1+r)^{-T}$. That is, given S, K, r, and T, there is an explicit relationship between European call and put values that must exist, regardless of market expectations. Note, however, that investors' bullish or bearish beliefs will influence S. But, given S, C cannot rise or fall relative to P.

In subsequent put–call parity propositions we introduce unknown dividends and/or American options. In these cases, boundaries are created within which $C-P$ can lie without permitting arbitrage. Within these boundaries, call prices can rise somewhat relative to put prices when investors are bullish, and put prices can rise somewhat rise relative to call prices when investors are bearish. The term "somewhat" is used because there are still put–call pricing restrictions even for options of these types.

16.4.2 European Options on Stocks That Pay Unknown Dividends

If the cash flows of the underlying assets are unknown, put–call parity becomes an inequality, and option prices can lie within a range of values that would preclude arbitrage.

Proposition XII: $S - \underline{D1}(1+r)^{-t1} - \underline{D2}(1+r)^{-t2} - K(1+r)^{-T} \geq C - P$
$\geq S - \overline{D1}(1+r)^{-t1} - \overline{D2}(1+r)^{-t2} - K(1+r)^{-T}$

In Proposition XII, $\underline{D1}$ and $\underline{D2}$ are the smallest dividends the arbitrageur believes possible at times $t1$ and $t2$. $\overline{D1}$ and $\overline{D2}$ are the largest dividends possible at those dates.[13] The present values of the smallest dividend amounts possible are subtracted from the price of the underlying asset on the left-hand side (LHS) of the inequality, while the present values of the largest possible dividends are subtracted from the price of the underlying asset on the right-hand side (RHS). You are encouraged to prove the RHS. The proof of the LHS proceeds as follows:

What if $S-\underline{D1}(1+r)^{-t1}-\underline{D2}(1+r)^{-t2}-K(1+r)^{-T}<C-P$
Then $S-\underline{D1}(1+r)^{-t1}-\underline{D2}(1+r)^{-t2}-K(1+r)^{-T}-C+P<0$
Or $-S+\underline{D1}(1+r)^{-t1}+\underline{D2}(1+r)^{-t2}+K(1+r)^{-T}+C-P>0$

				At Expiration	
Today		**Time $t1$**	**Time $t2$**	$S_T<K$	$S_T>K$
Buy stock	$-S$	$+D1$(actual)	$+D2$(actual)	$+S_T$	$+S_T$
Borrow	$+\underline{D1}(1+r)^{-t1}$	$-\underline{D1}$			
Borrow	$+\underline{D2}(1+r)^{-t2}$		$-\underline{D2}$		
Borrow	$+K(1+r)^{-T}$			$-K$	$-K$
Sell call	$+C$			0	$-(S_T-K)$
Buy put	$-P$			$+(K-S_T)$	0
	>0	≥ 0	≥ 0	0	0

Note that the *borrowings* include the present value of the *smallest* dividend believed possible at the *farthest* ex-dividend dates believed possible. In proving the RHS of Proposition XII, one *lends* the present value of the *largest* dividend believed possible at the *nearest* ex-dates believed possible.

16.4.3 American Options on Non-Dividend-Paying Stocks

For American options, put–call parity pricing restrictions become wider, and the proofs become more complicated, since either a put or a call must always be written (and thus exposed to possible early exercise). If a call is written on a stock that does not pay dividends, or if the stock will not trade ex-dividend before the options' expiration date, early exercise is not a factor (under our assumptions). But early exercise of in-the-money American puts is always a possibility, and the arbitrage proofs must account for this when the put is written (see the proof of the RHS of Proposition XIII shortly).

Proposition XIII: $S-K(1+r)^{-T}\geq C-P\geq S-K$

Proof of the LHS of Proposition XIII proceeds as follows:

What if $S-K(1+r)^{-T}<C-P$
Then $S-K(1+r)^{-T}-C+P<0$
Or $-S+K(1+r)^{-T}+C-P>0$

		At Expiration	
Today		$S_T<K$	$S_T>K$
Buy stock	$-S$	$+S_T$	$+S_T$
Borrow	$+K(1+r)^{-T}$	$-K$	
Write call	$+C$	0	$-(S_T-K)$
Buy put	$-P$	$+(K-S_T)$	0
	>0	0	0

Because there are no dividends, we do not have to guard against the possibility of the call's early exercise. However, this possibility cannot be ignored in the following proof of the RHS of Proposition XIII.

What if $C-P<S-K$

Then $C-P-S+K<0$

Or $-C+P+S-K>0$

Today		**If the Put Could Be Assigned Before *T***	**At Expiration (if the put was not assigned earlier)** $S_T<K$	$S_T>K$
Buy call	$-C$	+ Time value (if any)[14]	0	$+(S_T-K)$
Sell put	$+P$	Assigned: long stock; pay K	$-(K-S_T)$	0
Sell stock	$+S$	The short position offsets the long position	$-S_T$	$-S_T$
Lend	$-K$	$+K$ + interest	$+K$ + interest	$+K$ + interest
	>0	≥ 0	>0	>0

The key feature to the proof of the RHS is to guard against the possibility that the stock could plummet in value the day after the arbitrage transactions are made. In this proof, at the very least, the arbitrageur will earn one day's interest if the put is exercised early one day after the initial trades. This guarantees that there will never be a cash outflow at any date.

Note that is important that the loan of $\$K$ be made only on an overnight basis and be renewed daily. Otherwise, if the stock should plummet at the same time that interest rates soared, the call might be worthless if the put were exercised early, and the value of the loan might be less than $\$K$. For example, suppose that the arbitrage requires a loan of $K=\$40$ and that this is made for a period of $T=30$ days at an annual rate of 10%. The loan contract stipulates that $40(1.1)^{0.08219}=40.3146$ will be repaid one month hence. Then, the next day, interest rates rise to an annual rate of 12% and the put is assigned. When trying to sell the loan contract, the arbitrageur will find that other investors who will be willing to pay only the present value of \$40.3146, at the prevailing interest rate of 12%. With 29 days remaining, the selling value is then only $40.3146(1.12)^{-0.07945}$ $=\$39.9532$. If the call was worthless, the cash flow on the early exercise date would be negative. By lending one day at a time, positive interest will always be earned.

16.4.4 American Options on Stocks That Pay Unknown Dividends

To enforce the following proposition, arbitrageurs must be sure that nonnegative cash flows are earned at all times, including those just before and after any ex-dividend dates. In performing a conversion, a written call could be exercised just before any ex-dates. If the arbitrageur performs a reversal, then the written put can be assigned anytime (as early as "tomorrow"). To make the proof of proposition XIV easier, we will assume that only one ex-dividend day exists prior to the options' expiration date.[15]

Proposition XIV: $S - K(1+r)^{-T} \geq C - P \geq S - \overline{D1}(1+r)^{-t1} - K$

Proof of the LHS of Proposition XIV proceeds as follows:

What if $S - K(1+r)^{-T} < C - P$

Then $S - K(1+r)^{-T} - C + P < 0$

Or $-S + K(1+r)^{-T} + C - P > 0$

Today		At Time $t1$ the Call Could Be Assigned	If the Call Is Not Assigned Early at $t1$
Buy stock	$-S$	Deliver stock to the exerciser of the call	$+D1$
Borrow	$+K(1+r)^{-T}$	$-K(1+r)^{-T}$ – interest	
Sell call	$+C$	$+K$	
Buy put	$-P$	+ Time value (if any)	
	>0	>0	>0

If the call is exercised an instant before the first ex-dividend date, time $t1$, the arbitrageur delivers the stock and receives payment of \$$K$. The receipt of \$$K$ is sufficient to pay off the loan plus interest on the loan. The put is sold if it has any value. If the call is not assigned early, the arbitrageur receives the dividend.[16]

If the call is not exercised at time $t1$, and if there is only one ex-dividend day, only the expiration day needs to be considered to complete the proof. This was done for the LHS of Proposition XIV. Thus, the LHS of Proposition XIV is complete, and we present the proof of the RHS.

What if $C - P < S - \overline{D1}(1+r)^{-t1} - K$

Then $C - P - S + \overline{D1}(1+r)^{-t1} + K < 0$

Or $-C + P + S - \overline{D1}(1+r)^{-t1} - K > 0$

Today		(1) The Put Could Be Assigned Before $t1$	(2) If the Put Is Not Assigned at Time $t1$	(3) The Put Could Be Assigned After $t1$ but Before T	(4) If the Put Has Not Been Assigned by Time T	
					$S_T < K$	$S_T > K$
Buy call	$-C$	+Time value (if any)		+Time value (if any)	0	$+(S_T - K)$

Sell put	$+P$	Put assigned: acquire stock and pay K		Put assigned: acquire stock and pay K	$-(K-S_T)$	0
Sell stock	$+S$	Long position offsets short position	$-D1$(actual)	Long position offsets short position	$-S_T$	$-S_T$
Lend	$-\overline{D1}(1+r)^{-t1}$	$+\overline{D1}(1+r)^{-t1}$ +interest	$+\overline{D1}$			
Lend	$-K$	$+K+$interest		$+K+$interest	$+K+$int	$+K+$int
	>0	>0	≥ 0	>0	>0	>0

EXAMPLE 16.6 Consider the following actual market data for GM at 3 P.M. on February 15, 2000:

$S = 74\ ^1/_2$
$K = 75$
$C = 9$ for the Sept 75 call
$P = 8\ ^3/_4$ for the Sept 75 put
$r = 6\%$ per year
$T = 221$ days $= 0.6055$ year
$K(1+r)^{-T} = 75(1.06)^{-0.6055} = 72.40$

The last ex-dividend date was on February 7, 2000, and the dividend amount was \$0.50 per share. Thus, the arbitrageur might assume:

$\overline{D1} = 0.55$
$t1 = 90$ days $= 0.2466$ year
$\overline{D1}(1.06)^{-0.2466} = 0.5422$
$\overline{D2} = 0.60$
$t2 = 182$ days $= 0.4986$ year
$\overline{D2}(1.06)^{-0.4986} = 0.5828$

When there are two ex-dividend days prior to expiration, Proposition XIV states that:

$$S - K(1+r)^{-T} \geq C - P \geq S - \overline{D1}(1+r)^{-t1} - \overline{D2}(1+r)^{-t2} - K$$

$$74.5 - 72.4 \geq 9 - 8.75 \geq 74.5 - 0.5422 - 0.5828 - 75$$

$$2.1 \geq 0.25 \geq -1.625$$

As you can see, when dividends are substantial, and the American options have a long time to expiration, the put–call parity pricing restriction is not very strong. Look closely at Proposition XIV. As the dividends are paid off, and as T shrinks, the range of prices implied by the proposition will shrink. Here, market expectations will likely affect option values. If investors are bullish, we might observe $C-P$ to be about 2.1. If investors are bearish, we might see $C-P$ to be about -1.625.

The put, if it is in the money, could be exercised as early as tomorrow. Even if the call is worthless tomorrow, the arbitrageur still will earn one day's interest on the two loans made. This is demonstrated in column (1). If the put is not exercised before the next ex-dividend date, the arbitrageur pays the actual dividend due on the short sale of stock, but receives $\overline{D1}$. Then there is a nonnegative cash flow at time $t1$, as shown in column 2.

Then, the arbitrageur must ask, What if the written put is assigned after time $t1$, but before expiration? As shown in column 3, he is again assured of a positive cash inflow. Finally, column 4 presents the positive cash flows the arbitrageur receives if the written put is never assigned before it expires. This completes the proof of Proposition XIV.

The arbitrageur could take some risks (though by definition, he would no longer be an arbitrageur), and narrow the bounds he believes should exist around put and call prices. For example, he might assess the probability of early exercise of the call at time $t1$ to be so small that it can be safely ignored. Or he might believe there is little probability that the stock will fall so much that the put will be exercised in the next month. These assumptions will narrow the bounds implied by American put–call parity on stocks paying unknown dividends.[17]

16.5 Box Spreads Using European Options

From Proposition XI, we know $C-P=S-K(1+r)^{-T}$. Given the price of an underlying security, Proposition XI holds for all put–call pairs with the same time to expiration. As such, define $C(K1)$ and $P(K1)$ as a call and a put with a low strike price, $K1$. Further, define $C(K2)$ and $P(K2)$ as a call and a put with a higher strike price, $K2$. Thus,

$$C(K1) - P(K1) = S - K1(1+r)^{-T} \tag{16.8}$$

and

$$C(K2) - P(K2) = S - K2(1+r)^{-T} \tag{16.9}$$

Subtracting Equation (16.9) from Equation (16.8) yields:

$$C(K1) - C(K2) - P(K1) + P(K2) = (K2 - K1)(1+r)^{-T} \tag{16.10}$$

Note that the stock price does not appear in (16.10). To prove that equation (16.10) must hold, we proceed with a two-part proof.[18]

What if $C(K1)-C(K2)-P(K1)+P(K2)>(K2-K1)(1+r)^{-T}$

Then $C(K1)-C(K2)-P(K1)+P(K2)-(K2-K1)(1+r)^{-T}>0$

		At Expiration		
Today		$S_T<K1$	$K1<S_T<K2$	$K2<S_T$
Sell call	$+C(K1)$	0	$-(S_T-K1)$	$-(S_T-K1)$
Buy call	$-C(K2)$	0	0	$+(S_T-K2)$
Buy put	$-P(K1)$	$+(K1-S_T)$	0	0

Sell put	$+P(K2)$	$-(K2-S_T)$	$-(K2-S_T)$	0
Lend	$-(K2-K1)(1+r)^{-T}$	$+(K2-K1)$	$+(K2-K1)$	$+(K2-K1)$
	>0	0	0	0

The four option positions and lending today generates a cash inflow. This is because the call with the lower strike price, $K1$, will have more intrinsic value than a call with the higher strike price, $K2$. Therefore, the call with the lower strike price is more valuable. Similarly, the put with the higher strike price, $K2$, will be more valuable than the put with the lower strike price. In the end-of-chapter problems, you are asked to prove these two relationships.

Note that the underlying stock price does not appear in Equation (16.10). In words, Equation (16.10) says that a risk-free asset is created by selling a call with strike $K1$, buying a call with strike $K2$, buying a put with strike $K1$, and selling a put with strike $K2$.[19] This option combination is known as a "short box spread." One lends risklessly by purchasing a call with strike $K1$, selling a call with strike $K2$, selling a put with strike $K1$, and buying a put with strike $K2$. This position is known as a "long box spread." The box spread is a popular and convenient way for option traders to borrow and lend through the options markets.

16.6 Summary

In this chapter, we established certain relationships concerning the pricing of calls relative to the underlying stock price, relative to time to expiration, and relative to the calls' strike prices. Then, analogous boundary relationships were derived for puts.

By mastering the techniques for deriving the propositions, you should now be able to understand the process of arbitrage. If markets are working well, there should not be any arbitrage opportunities. However, some investors, such as traders on the floors of the different exchanges, have such low transactions costs that they can buy at the bid price and sell at the asked price, hence they may be able to arbitrage. Arbitrage is the price-setting mechanism in options markets as well as in the financial futures markets.

From these pricing restrictions, we were also able to establish that an American call on a non-dividend-paying stock would never be exercised early. The only time an American call would be exercised early is on the day before the stock trades ex-dividend. On the other hand, in-the-money American puts could frequently be exercised early.

Finally, we established put–call parity, which creates no-arbitrage pricing relationships between puts and calls. The propositions differ, depending on whether the options are American or European, and on whether the underlying asset will have a cash distribution such as a dividend prior to the options' expiration.

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Notes

[1]Many of the results of this chapter were originally stated in Merton (1973). Further development occurred in Cox and Rubinstein (1985, Chapter 4) and Jarrow and Rudd (1983, Chapters 4–6).

[2]Note that there arbitrageurs who own the stock *prior* to the arbitrage trades might exist. Rather than selling the stock short, they can just sell the stock they own. For them, selling actually does yield full use of the proceeds from the required sale. Sometimes this is referred to as "quasi-arbitrage." Quasi-arbitrageurs switch one investment for another that dominates the original.

[3]The terms "exercised" and "assigned" are used interchangeably in this book. However, "assigned" usually refers to what is experienced by the option writer. The option owner exercises her option. The writer is assigned the exercise.

[4]Floor traders frequently exploit violations of Proposition II' when there is only a short time to expiration. Their bid prices will often be 1/8 or 1/4 below the intrinsic values of in-the-money options.

[5]Whenever long-term capital gains are taxed at a lower marginal rate than short-term capital gains, investors must consider the trade-offs of a realizing a certain short-term capital gain versus holding onto a risky investment to (hopefully) realize a long-term capital gain because the latter would be taxed at a lower rate. If an investor exercises a call, the holding period for the stock begins on the day after the exercise date. Thus, in early December, an investor might have a large unrealized profit on a call that expires in December. If she sells it, she would have to pay taxes on the profit at her high marginal tax rate. However, if she was still bullish on the stock, she might exercise it and (a) defer the taxes on the profit until the following year, and (b) by holding the stock for the required period (perhaps six months or a year), possibly stretch the short-term gain into a long-term gain. By exercising the call early, the investor would begin the holding period earlier.

[6]If there were two ex-days the proposition would read:

$$C \geq \max[0, S - \overline{D1}(1+r)^{-t1} - \overline{D2}(1+r)^{-t2} - K(1+r)^{-T}].$$

$\overline{D1}$ is the greatest dividend that will be paid at the next ex-date, and $\overline{D2}$ is the maximum dividend at the following ex-date. In effect, think of stock value as consisting of two components: the present value of dividends that will be paid between today and the option's expiration date, and the present value of all dividends subsequent to the expiration date. When you buy a European call, you are buying a call on the latter only. Thus, in setting the arbitrage boundary, the dividends that are to be paid prior to expiration are subtracted from the stock's current value.

[7]See Smith (1976, pp. 13–14) and Roll (1977) for additional discussion on the early exercise of American calls on dividend-paying stocks.

[8]For the case of two dividends, Proposition IX would be as follows:

$$P \geq \max[0, K(1+r)^{-T} - S + \underline{D1}(1+r)^{-t1} + \underline{D2}(1+r)^{-t2}]$$

[9]The example presented in this paragraph is different from the methods of the proofs in that a zero cash flow at time zero, and time $t1$, is achieved, and the arbitrage profit is realized at time $t2$. The setup could have been presented just as easily to realize the profit at the initiation of the trades by borrowing the present values of the dividends and also borrowing the present value of the strike price; you are encouraged to work it out in this way, too.

[10]Market makers will buy the deep in-the-money American puts, but only at bid prices below intrinsic value; they will then earn arbitrage profits.

[11]This is demonstrated by Geske and Shastri (1985).

[12]Leverage is a double-edged sword. If the stock price was to decline to $29/share,which is a mere 3.33% decline in value, both the call owner and the buyer of stock on margin would suffer much greater percentage declines in wealth.

[13]If the underlying asset is paying a *known* dividend amount, Proposition XII collapses to be $C-P=S-D1(1+r)^{-t1}-D2(1+r)^{-t2}-K(1+r)^{-T}$. You are also encouraged to prove this version of the proposition.

[14]The put will be exercised early only if it is in the money. Thus the call will not have any intrinsic value.

[15]If the underlying asset pays a known dividend, then Proposition XIV becomes $S-K(1+r)^{-T}\geq C-P\geq S-PV(\text{divs})-K$, and PV(divs) = present value of all dividends that will be paid prior to the options' expiration day.

[16]If there were two dividends, the same logic would apply on day $t2$. Should the written call be assigned early, the receipt of $K still will suffice to repay the loan plus interest.

[17]If interest rates are known at every date, the bounds for American put–call parity can be narrowed. See Jarrow and Rudd (1983, pp. 72–75).

[18]Part B of the proof simply reverses all the option positions and includes borrowings. We leave this as an exercise for the reader.

[19]One can also begin this proof by using Proposition XII. You are encouraged to do so.

PROBLEMS

16.1 Suppose there is only one ex-dividend date between today and the expiration day of a call. State Proposition IV.

16.2 Suppose there are three ex-dividend dates between today and the expiration day of a call. State Proposition IV.

16.3 Prove Proposition IVc.

16.4 Prove Proposition Xd.

16.5 Discuss the factors that increase the likelihood that an American put will be exercised early.

16.6 Discuss the factors that increase the likelihood that an American call will be exercised early.

16.7 Why should an American call never be exercised early, except perhaps just before it trades ex-dividend?

16.8 Given the following data, what is the lowest price of an American call?

$S=55$

$K=50$

$T=3$ months

$r=10\%$/year

The next ex-dividend date will be two months hence. You have established the following probabilities for the dividend amount:

Probability	Dividend Amount
0.01	$0.15
0.10	$0.17
0.50	$0.19
0.38	$0.21
0.01	$0.23

16.9 Assume that $S=44$, and use all the other data in Problem 16.8. Compute the lowest value of an American put.

16.10 Given the following information, should the call be exercised early?

$S=60$

$K=50$

$C=10$
$T=2$ months
$r=10\%$/year

The stock will trade ex-dividend tomorrow. The dividend amount will be \$1/share. You believe that the stock price will decline by the dividend amount.

16.11 Is the following statement true, false, or uncertain? "The time value of any American option is never negative. The time value of a European option might be negative." Provide an explanation for your answer.

16.12 Discuss why an American put will increase in value the longer the time to expiration, but a European put can increase or decrease in value as time to expiration lengthens.

16.13 Suppose you are completely indifferent between exercising early a call that you own and holding onto it for another day. Tomorrow, the underlying stock trades ex-dividend. You are fully knowledgeable about when, if at all, to rationally exercise your call. Then, you receive new information: interest rates have just sharply risen. Given that you were previously indifferent about exercising your call early, and now, interest rates are higher than they were before, would the interest rate rise cause you to exercise early, or to hold onto the call for another day? Explain your answer.

16.14 Suppose that on February 2, 2001, ZZZ Corp. common stock closes at 39.40, and the April 45 put option on ZZZ closes at 5.50. You believe that ZZZ will trade ex-dividend on March 21. Also, you have formed the following probabilities concerning the possible dividend amounts:

Probability	Amount per Share
0.01	\$0.10
0.80	\$0.30
0.18	\$0.32
0.01	\$0.35

Assume that the riskless interest rate is 8%. Do the April 45 ZZZ puts violate Proposition X?

16.15 Today is the last with-dividend day for the common stock of Miss Molly's Buggywhips (MMB). Today's stock price is \$47/share. Tomorrow, MMB trades ex-dividend. The dividend that will be paid is one dollar. On the ex-dividend day, you expect MMB's stock price to decline by the dividend amount.

You own a call option with a strike price of 40. It has two months until expiration. The riskless interest rate is 10%/year.

a. If the call is selling at its lower bound today, what is its premium?

b. If you expect the call to sell at its lower bound tomorrow, what will be its premium tomorrow?

c. Based on your answers to parts a and b, is it rational and optimal for you to exercise your call? Why or why not?

16.16 Suppose $S=40$ and $r=10\%$/year. The stock pays no dividends. A European call with a strike of 40 and time to expiration of 6 months sells for 7. A European put with the same K and T sells for 4. What riskless rate of return can an investor realize over the six-month period by writing the call, buying the put, and buying the stock?

16.17 Prepare a profit diagram for each of the following two strategies: (a) buy a put with a strike price of 45 and (b) buy a call with a strike of 45 and sell 100 shares of the stock short. The put is selling for 3.8, the call is selling for 1.5, and the stock sells for 42.5. Discuss the similarities and the differences between the two profit diagrams.

16.18 Define $C(T1)$ as a call expiring at nearby date $T1$, and $C(T2)$ as a call expiring at a more distant date, $T2$. $C(T1)$ and $C(T2)$ have the same underlying asset and the same strike price. If the underlying asset will not trade ex-dividend before time $T2$, or if the calls are American, prove that $C(T2) \geq C(T1)$. To get you started on this question:

What if	$C(T2) < C(T1)$
Then	$C(T2) - C(T1) < 0$
Or	$-C(T2) + C(T1) > 0$

a. Assume that at any date before the first expiration date, $T1$, the nearby American call could be assigned. Today, you would buy $C(T2)$ and sell $C(T1)$. What trades do you make if you are assigned $C(T1)$?

b. Assume that the first expiration date, T1, comes and goes but the nearby American call was not assigned. What trades do you make at $T2$?

c. Now, define $P(T1)$ as the price of an American put expiring at nearby time $T1$ and $P(T2)$ as the price of a put expiring at a more distant date $T2$. Each put has the same underlying asset and the same strike price. Prove that $P(T2) \geq P(T1)$. In your proof, make sure you account for the possibility that a put could be assigned.

16.19 Suppose:

$S = 20$

$K = 20$

$T1$ = expiration date for nearby call = 2 months

$T2$ = expiration date for distant call = 5 months

$t1$ = next ex-dividend date = 3 months

Under what conditions would it be possible for the short-term European call to sell for *more* than a longer term European call? How about European puts? Can a long-term European put sell for less than an otherwise identical European put with less time to expiration?

16.20 Define $C(K1)$ as an American call with a low strike price $K1$, and $C(K2)$ as an American call with a higher strike price $K2$. Each call has the same underlying asset and the same time to expiration. Prove that $C(K1) \geq C(K2)$. To get you started:

What if	$C(K1) < C(K2)$
Then	$C(K1) - C(K2) < 0$
Or	$-C(K1) + C(K2) > 0$

a. As an additional hint, today you would buy $C(K1)$ and sell $C(K2)$. Assume that the written American call could be assigned at any date before the expiration date. What trades do you make if you are assigned $C(K2)$?

b. Assume that the expiration date comes and goes, but the written American call was not assigned. What trades do you make at expiration?

c. Suppose the calls are European. Can you still prove that $C(K1) \geq C(K2)$?

d. Suppose the stock pays dividends. Can you still prove that $C(K1) \geq C(K2)$?

e. Now, define $P(K1)$ as an American put with a low strike price $K1$, and $P(K2)$ as an American put with a higher strike price $K2$. Each put has the same underlying asset and the same time to expiration. Prove that $P(K2) \geq P(K1)$. In your proof, make sure you account for the possibility that a put could be assigned.

16.21 Consider an American put with a strike price of 65. The price of the put is 4. The price of the underlying asset is 60. Which of the following would you do to arbitrage?

a. Buy the put, buy the underlying asset, exercise the put.

b. Buy the put, sell the underlying asset, exercise the put.

c. Sell the put, buy the underlying asset, exercise the put.

d. Sell the put, sell the underlying asset, exercise the put.

CHAPTER 17

The Binomial Option Pricing Model

At expiration, we know that an option is worth its intrinsic value. Thus, at expiration, an option's intrinsic value serves as an exact option pricing model. However, it is also important to be able to value options before expiration. While the put–call parity relationship does generate put and call option price boundaries before expiration, an important condition of the put–call parity relationship is that a put price must be known before a call price can be generated. Thus, although put–call parity and other pricing bounds shown in Chapter 16 narrowed the range of possible option prices, we still do not have a model that provides an exact option price before expiration.

In this chapter, we derive a model that provides an exact option price before expiration. This model is called the binomial option pricing model (BOPM). One of the advantages of this particular model is its flexibility.[1] The binomial option pricing model can be used to value complex options of many types, including American puts, American calls on dividend-paying stocks, options on debt instruments and interest rates, and exotic options. It can also accommodate changing conditions over time. For example, if interest rates are believed likely to change during the life of the option, or if the volatility of the underlying asset is believed likely to change, the BOPM can handle these situations quite easily. Even more, the BOPM can account for different assumed stock price movement processes, such as one in which the variance of the stock's returns is greater at lower price levels (constant elasticity of variance), or if there is a probability of a "jump" (a discontinuity, perhaps due to a possible tender offer for the stock) in the stock's price at each date.[2]

Studying the BOPM in detail also allows us to achieve additional understanding of several aspects of options that we have already learned. For example, the BOPM clarifies conditions under which options will be exercised early. Moreover, it validates the concept that buying a call is like buying stock and borrowing, and the concept that buying a put is like selling stock short and lending. The idea that options can be replicated with portfolios of stocks and bonds, which has revolutionized the field of finance, permits institutions to hedge their option portfolios. Option replication gave rise to the portfolio insurance industry that had an estimated size of $80 billion in September 1987 and has even been blamed for the stock market crash of October 1987.[3] Finally, under certain assumptions, the BOPM can be shown to converge to several different option pricing models. These include the Black–Scholes option pricing model (BSOPM), which is the most widely used option pricing model, and the subject of the next chapter.

17.1 A QUIZ

Before beginning with the details of the BOPM, take about 30 minutes and answer the following questions. Doing so will make the subsequent material easier to follow. The answers are at the end of the quiz, but try to work through the questions independently.

Consider a European call option on a stock. The call has one period (say one month) until expiration. The current stock price, S_{T-1}, is \$40. At the expiration date, time T, the stock will sell for one of only two possible prices: either $S_{T,u} = \$50$, or for $S_{T,d} = \$35$. The strike price is \$45.

1. If the stock rises to \$50 on the expiration date, what is the price of the call option? If $S_{T,u} = (1+u)S_{T-1}$, then u is the rate of return on the stock if there is an uptick. What is the value of u?
2. If the stock falls to \$35 at time T, what will the call be worth at that time? If $S_{T,d}=(1+d)\times S_{T-1}$, what is the value of d?

As will soon be discussed, we can create a levered portfolio (consisting of a long position in stock, financed in part by borrowed funds) that will duplicate the payoffs of the call. Suppose the riskless interest rate is $r=1\%$ per period (e.g., 1% per month) and that investors can borrow and lend at that rate. Consider a portfolio, created at time $T-1$, consisting of a third of a share of stock and \$11.551155 in borrowed funds. This is a levered portfolio financed at the riskless interest rate.

3. If the stock were to rise to \$50, how much would the levered portfolio be worth at time T? Be sure to pay the interest on the borrowed funds.
4. If the stock were to fall to \$35 at time T, what would the levered portfolio be worth?

Your answers to 3 and 4 should be identical to the call values you computed in 1 and 2. The call offers the exact same payoffs (at time T) as the levered portfolio. Therefore, at time $T-1$, the call should sell for the same price as the amount invested in the levered portfolio. If it did not, all investors would buy the cheaper assets and sell short the more expensive ones.

5. How much of the investor's money was invested in the levered portfolio at time $T-1$? This must also equal the price of the call at time $T-1$. In other words:

$$0.333333S_{T-1} - 11.551155 = C_{T-1}$$

The left-hand side expresses the amount invested in the levered portfolio: a third of a share of stock, and \$11.551155 in borrowed funds.

6. If the call sells for \$2 at time $T-1$, how can an investor earn an arbitrage profit? Prepare an arbitrage table like those shown in Chapter 16 by finding the positive cash inflow at time $T-1$ and showing that no matter what S_T is, there is a zero cash flow at time T.
7. If the call sells for \$1.50 at time $T-1$, how can an investor earn an arbitrage profit? Prepare an arbitrage table like those in Chapter 16 by finding the positive cash inflow at time $T-1$, and showing that no matter what S_T is, there is a zero cash flow at time T.
8. Would your answer to any of the foregoing questions change in any way if you knew that the probability of the stock rising was 90%? Would your answers change if the probability was only 5% that the stock would rise?
9. It is generally agreed that arbitrage cannot exist in well-functioning markets, simply because investors maximize wealth. That is, they prefer more to less. In other words, if an arbitrage opportunity came along, investors would quickly exploit it to the maximum extent possible. Do any of the foregoing results depend on the level of investors' risk aversion? Do they change if investors are risk neutral or risk seekers?

You might be wondering how the third of a share of stock and the \$11.551155 in debt were chosen. They were computed because they are the unique numbers that equated the payoffs of the call with the payoffs of the levered portfolio, and they will be algebraically defined by a set of equations in the next section.

The answers to the quiz are presented next. However to get maximum benefit out of the exercise, you should work out the solutions before proceeding with the rest of the chapter.

Answers to the Quiz

1. $C_{T,u}=5$; $u=0.25$
2. $C_{T,d}=0$; $d=-0.125$
3. $50(0.333333)-11.551155(1.01)=5$
4. $35(0.333333)-11.551155(1.01)=0$
5. $S_{T-1}/3-11.551155=C_{T-1}=1.782178$, given $S_{T-1}=40$

6.

		Time T	
Time $T-1$		$S_T=35$	$S_T=50$
Sell call	+2	0	−5
Buy $^1/_3$ share of stock	−13.333333	+11.66667	+16.66667
Borrow	+11.551155	−11.66667	−11.66667
	+0.217822	0	0

7.

		Time T	
Time $T-1$		$S_T=35$	$S_T=50$
Buy call	−1.50	0	+5
Sell $^1/_3$ share of stock	+13.333333	−11.66667	−16.66667
Lend	−11.551155	+11.66667	+11.66667
	+0.282178	0	0

8. No
9. No

17.2 Deriving the Binomial Option Pricing Model for Calls on Non-Dividend-Paying Stocks

17.2.1 Assumptions

To begin, we assert that investors prefer more to less. This assertion is important because we can be sure that all arbitrage opportunities will be instantaneously exploited under this assertion. In addition, we make the following sets of assumptions.

1. Markets are perfect and competitive. This means that there are no transactions costs, no margin requirements, and no taxes. In addition investors receive full use of the proceeds from short sales, and, if they wish, investors can trade fractions of securities (i.e., securities are infinitely divisible). Also they can trade all they want at the market prices at every date. Finally, there is only one interest rate, r, and investors can risklessly borrow and lend at that rate.

2. The periodic interest rate r, and the sizes of the uptick u and downtick d are known in every future period. The stock can move only according to this "geometric random walk" (up by u% per period, or down by d% per period). It is not necessary for u, d, and r to be constant in every period. It is only necessary for them to be known. In other words, u, d, and r are deterministic.

Note that there are some "assumptions" that you might think are necessary. For example, there are no assumptions about the expected or required return on the stock. Also, there are no assumptions about the investors' degree of risk aversion, if any. Moreover, nothing is assumed about the probability of the uptick or the downtick. Note well, however, that we do assume that the pricing process of the underlying asset is known. That is, we assume that we know the sizes of u and d in each period.

Before we derive the single-period BOPM in the next section, we present an overview of how call option prices are calculated using the BOPM. Today is time $T-1$, and the option expires at the end of the single period, i.e., at time T. To calculate a call option price today using the BOPM, follow these steps:

1. Define the stock price process. Given today's price, the stock can take on one of only two possible prices at the next date: $(1+u)$ times the current stock price, or $(1+d)$ times the current stock price.

2. Determine the terminal prices of a call option. There will be two possible call prices, one for each possible stock price. These prices are equal to the intrinsic value of the call option at expiration.

3. Equate the payoffs of an unknown "equivalent portfolio" of stocks and bonds with the payoffs of the call. There will be two equations (one for each stock price) and two unknowns (the number of shares of stock to be bought and the amount of debt). Thus, the composition of the equivalent portfolio can be determined.

4. Apply the law of one price: if two assets offer identical payoffs for every possible future outcome (here, there are two possible outcomes that will occur at the next date: the stock will change by either u% or d%), then today's prices of the two assets must be equal. Since the equivalent portfolio and the call offer the same payoffs, they must be worth the same today.

17.2.2 The Derivation of the One-Period Binomial Option Pricing Model

We start with a one-period, two-date model. At time $T-1$, the stock price is S_{T-1}. At the option's expiration date, time T, the stock price can take on one of only two values. If there is an uptick at time T, then $S_{T,u}=(1+u)S_{T-1}$. If there is a downtick at time T, then the stock will take on the lower value, $S_{T,d}=(1+d)S_{T-1}$. Graphically, this is depicted as follows:

$$S_{T-1} \begin{cases} S_{T,u} = (1+u)S_{T-1} \\ S_{T,d} = (1+d)S_{T-1} \end{cases}$$

The parameters u and d are the possible rates of return on the underlying asset. For stocks, it is realistic to expect u to be greater than zero and d to be less than zero. However, this might not be the case for all underlying assets. For example, if pure discount Treasury bill prices were being modeled, it might be more realistic to have both u and d greater than zero.

The parameter r equals the riskless interest rate during the single period. To preclude arbitrage opportunities, u must exceed r and r must exceed d. If u and d were both greater than r. investors could arbitrage by borrowing as much as possible to invest in the risky stock. At time T, it would then be certain that the risky stock price will be high enough to repay the loan and still have some cash left over. In other words, if $u > d > r$, then

	Time T	
Time $T-1$	**Uptick**	**Downtick**
Borrow \$1	$-\$1(1+r)$	$-\$1(1+r)$
Invest \$1 in the stock	$+\$1(1+u)$	$+\$1(1+d)$
Net cash flow = \$0	$u-r>0$	$d-r>0$

Similarly, we cannot have $r > u > d$, for in this case, no rational person would ever buy the risky stock. After all, no matter what outcome occurs at time T, the investor would have been better off buying the riskless asset. Equivalently, we could demonstrate how an arbitrage profit could be earned by selling the stock short and using the proceeds to invest in riskless securities. Problem 17.1 at the end of the chapter illustrates the arbitrage process if $r > u > d$.

Returning to the binomial model, if time T is the expiration date of a call option with a strike price of K, the call will be worth $C_{T,u}=\max(0, S_{T,u}-K)$ if the stock price rises. If the stock price falls in value at time T, the call would be worth $C_{T,d}=\max(0, S_{T,d}-K)$. Graphically, we have

$$C_{T-1} \begin{cases} C_{T,u} = \max(0, S_{T,u}-K) = \max[0,(1+u)S_{T-1}-K] \\ C_{T,d} = \max(0, S_{T,d}-K) = \max[0,(1+d)S_{T-1}-K] \end{cases}$$

Now consider the possibility of forming a portfolio that will make the same payoffs at time T as the call. We will buy a fraction of a share of stock, denoted by Δ, and invest $\$B$ in riskless bonds (debt instruments). We will see that for a portfolio that is equivalent to a call, Δ will always be positive or zero (nonnegative) and B will always be negative or zero (nonpositive). This means that a call is equivalent to a portfolio consisting of a long position in stock and borrowing. This is a levered portfolio (i.e., a purchase of stock made on margin). On the other hand, for a put, Δ will always be nonpositive and B will always be nonnegative. Thus, a put is equivalent to a short position in the underlying asset, combined with lending.

The debt–equity equivalent portfolio will require an investment of $\Delta S_{T-1}+B$. In the case of a call, $B \leq 0$, $\Delta \geq 0$, and $\Delta S_{T-1} \geq |B|$.[4] The debt–equity portfolio will either be worth $\Delta S_{T,u}+(1+r)B$ if there is an uptick in the stock, or $\Delta S_{T,d}+(1+r)B$ if there is a downtick in the stock. Graphically, this is

$$\Delta S_{T-1} + B \begin{cases} \Delta(1+u)S_{T-1} + (1+r)B = \Delta S_{T,u} + (1+r)B \\ \Delta(1+d)S_{T-1} + (1+r)B = \Delta S_{T,d} + (1+r)B \end{cases}$$

Now, equate the payoffs of the equivalent portfolio with the values of the call at time T:

$$\Delta(1+u)S_{T-1} + (1+r)B = C_{T,u}$$
$$\Delta(1+d)S_{T-1} + (1+r)B = C_{T,d}$$

This is a set of two simultaneous equations with two unknowns, Δ and B. Our goal is to find the values of these unknowns, in which case we will have defined a levered portfolio that pays off the same as the call. Solving the system of simultaneous equations, we get:

$$\Delta = \frac{C_{T,u} - C_{T,d}}{(u-d)S_{T-1}} = \frac{C_{T,u} - C_{T,d}}{S_{T,u} - S_{T,d}}; \quad \Delta \geq 0 \tag{17.1}$$

$$B = \frac{(1+u)C_{T,d} - (1+d)C_{T,u}}{(u-d)(1+r)}; \quad B \leq 0 \tag{17.2}$$

The general formulation for Δ and B merely alters the notation of Equations (17.1) and (17.2):

$$\Delta = \frac{C_u - C_d}{(u-d)S} = \frac{C_u - C_d}{S_u - S_d} \tag{17.3}$$

$$B = \frac{(1+u)C_d - (1+d)C_u}{(u-d)(1+r)} \tag{17.4}$$

The value of Δ tells us the fraction of one share of stock to buy in order to replicate one call.[5] That is, $0 \leq \Delta \leq 1$. The value of B, where $B \leq 0$, specifies how much to borrow to finance the investment in the stock.

If the call and the debt–equity portfolio both offer exactly the same payoffs at time T, then the price of the call at time $T-1$ must equal the investment in the equivalent portfolio at time $T-1$:

$$C_{T-1} = \Delta S_{T-1} + B \tag{17.5}$$

The general statement that a call is equivalent to a unique levered position in the underlying asset is thus[6]:

$$C = \Delta S + B \tag{17.6}$$

If the call sold for less than the equivalent portfolio, an arbitrageur would buy the call and sell the equivalent portfolio (i.e., sell the stock and lend). This would lead to a positive cash inflow at time $T-1$, with a zero cash flow at time T, regardless of whether the stock rises or falls. Question 7 in the quiz at the start of the chapter illustrates this.

If the call sold for a price higher than $\Delta S_{T-1} + B$, an arbitrage profit could be earned by selling the call and buying the equivalent portfolio (buy the stock and borrow). Question 6 in the quiz illustrates this.[7]

Next, substitute the expressions for Δ [Equation (17.1)] and B [Equation (17.2)] into the expression for the call value [Equation (17.5)]. After simplifying, we get:

$$C_{T-1} = \frac{[(r-d)/(u-d)]C_{T,u} + [(u-r)/(u-d)]C_{T,d}}{1+r}$$

or

$$C_{T-1} = \frac{pC_{T,u} + (1-p)C_{T,d}}{1+r} \tag{17.7}$$

where

$$p = \frac{r-d}{u-d} \quad \text{and} \quad 1-p = \frac{u-r}{u-d}$$

From this, we derive the general binomial option pricing model equation for the value of a call option on a non-dividend-paying stock:

$$C = \frac{pC_u + (1-p)C_d}{1+r} \tag{17.8}$$

Equation (17.8) defines the BOPM single-period value of a call option. This must be the value of the call if we know the process of stock price movements (the u and the d), if we know the riskless interest rate, if we can trade without frictions such as commissions, if assets are infinitely divisible (i.e., we can trade fractions of assets), and if investors prefer more to less. Nowhere did we ever mention risk aversion. Investors can be extremely risk averse, not very risk averse, risk neutral, or even risk seekers, and we would get the same result. Furthermore, nowhere did we mention the probabilities of an uptick and downtick. It does not matter if investors assign a probability of 0.98 to an uptick, or if the probability of an uptick is 0.01.

If time T is not an ex-dividend date, it can be demonstrated that the formula for the call value in Equations (17.7) and (17.8), C_{T-1}, will always exceed or equal $S_{T-1}-K$. Thus, we do not have to worry that Equation (17.8) will produce a value that permits an arbitrage opportunity by violating Proposition II′ in Chapter 16. If it is certain that the call will finish in the money [i.e., $(1+d)S_{T-1}>K$], then $C_{T-1} \geq S_{T-1}-K$ (if time T is not an ex-dividend day).

17.2.3 Risk Neutrality

Looking at Equation (17.8), it is tempting to view p as a probability measure, in which case the value of the call equals the present value of the expected value of the call:

$$C_{T-1} = \frac{\begin{gathered}[(\text{probability of an uptick} \times \text{value of the call if the stock has an uptick}) \\ +(\text{probability of a downtick} \times \text{value of the call if the stock has a downtick})]\end{gathered}}{1+r}$$

Of course in deriving Equation (17.8), we never considered probabilities.

Nevertheless, the parameter p does have one element of being a probability measure, since $0 \leq p \leq 1$, and there is one interesting case in which p *is* a probability. That occurs when investors are *risk neutral*. If investors are risk neutral, they do not care about risk. All assets will be priced to yield the same risk-free rate of return. If any asset had an expected return above the riskless rate of interest, all investors would buy it, regardless of its risk (we are continuing to assume that investors prefer higher expected returns to lower ones).

Define q as the probability of an uptick. If investors are risk neutral, the stock (like any asset) must be priced to yield the riskless rate of return:

$$\begin{aligned} S_{T-1} &= \frac{q(1+u)S_{T-1} + (1-q)(1+d)S_{T-1}}{1+r} = \frac{qS_{T,u} + (1-q)S_{T,d}}{1+r} \\ &= \frac{\text{the expected stock price at time } T}{1+r} = \frac{E(S_T)}{1+r} \end{aligned}$$

Solving $S_{T-1}=[q(1+u)S_{T-1}+(1-q)(1+d)S_{T-1}]/(1+r)$ for q, we have:

$$q=\frac{r-d}{u-d}$$

This is the same value we found for p. In a risk-neutral economy, the call must also be priced to yield the riskless rate of return:

$$C_{T-1}=\frac{qC_{T,u}+(1-q)C_{T,d}}{1+r}=\frac{pC_{T,u}+(1-p)C_{T,d}}{1+r}=\frac{E(C_T)}{1+r}$$

Thus, the binomial option pricing model solution for the value of a call equals the value of a call in a risk-neutral economy. The parameter p is the probability of an uptick in an economy characterized by risk-neutral investors. However, keep in mind that p need not be the probability of an uptick if investors are risk averse. In any type of market environment, it will always be the case that $p=(r-d)/(u-d)$.

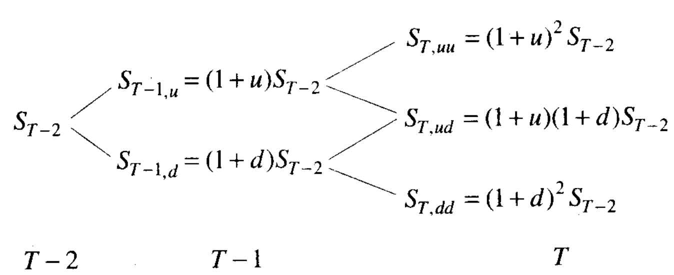
The idea that options are valued the same in a risk-neutral economy as in any economy has led to the simplification of what were previously very complicated solutions of option pricing problems.[8]

17.2.4 The Two-Period Binomial Option Pricing Model

Now we will extend the BOPM to a two-period world. We will assume that the process guiding the price of the underlying asset is stationary over time. This means that we assume that u and d are constant in every period. This assumption is *not* necessary to derive a multiperiod option model; u and d are assumed to be constant only to obtain a "nice" call option pricing formula for a multiperiod case.

In a two-period world, time progresses as shown in Figure 17.1. Time T is the expiration date of the call option. We continue to assume that there are no ex-dividend days.

The stock price process is:



The current stock price is S_{T-2}. At the end of the next period, time $T-1$, the stock price can take on one of two values, depending on whether there is an uptick or a downtick. At the end of the second period, time T, the stock price can be one of three values. Note that the time T stock price is the same regardless of whether an uptick follows a downtick or a downtick follows an uptick. This will always be the case if u and d are constants over time. Thus, there are two ways of realizing the stock price $S_{T,ud}$, which equals $S_{T,du}$.

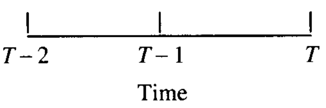


Figure 17.1

The pricing process of the call is:

$$C_{T-2} \to \begin{cases} C_{T-1,u} \to \begin{cases} C_{T,uu} = \max[0,(1+u)^2 S_{T-2} - K] \\ C_{T,ud} = \max[0,(1+u)(1+d)S_{T-2} - K] \end{cases} \\ C_{T-1,d} \to \begin{cases} C_{T,ud} = \max[0,(1+u)(1+d)S_{T-2} - K] \\ C_{T,dd} = \max[0,(1+d)^2 S_{T-2} - K] \end{cases} \end{cases}$$

$T-2 \qquad T-1 \qquad T$

We use a recursive process to solve for C_{T-2}. To do this, we start at the option expiration date and work our way back to the present. Thus, we can compute the value of $C_{T-1,u}$ by the one-period analysis we already learned. In other words, pretend we are at time $T-1$ and the stock just had an uptick. In this case, we are in the following situation:

$$S_{T-1,u} = (1+u)S_{T-2} \to \begin{cases} S_{T,uu} = (1+u)^2 S_{T-2} \\ S_{T,ud} = (1+u)(1+d)S_{T-2} \end{cases}$$

$$C_{T-1,u} \to \begin{cases} C_{T,uu} = \max[0,(1+u)^2 S_{T-2} - K] = \max[0,(1+u)S_{T-1,u} - K] \\ C_{T,ud} = \max[0,(1+u)(1+d)S_{T-2} - K] = \max[0,(1+d)S_{T-1,u} - K] \end{cases}$$

In this case, it is as if we are in a single-period world. Equation (17.9) defines the value of the call at time $T-1$, given that there was an uptick during the first period:

$$C_{T-1,u} = \frac{pC_{T,uu} + (1-p)C_{T,ud}}{1+r} \tag{17.9}$$

Equation (17.9) is just a restatement of the general single period BOPM call pricing model on a non-dividend-paying stock, Equation (17.8). The only difference is a slight change in the notation.

Now, let us assume instead that we are at time $T-1$, but instead of an uptick. there was just a downtick realized in the first period. Thus, the situation facing us is:

$$S_{T-1,d} = (1+d)S_{T-2} \to \begin{cases} S_{T,ud} = (1+u)(1+d)S_{T-2} \\ S_{T,dd} = (1+d)^2 S_{T-2} \end{cases}$$

$$C_{T-1,d} \begin{cases} C_{T,ud} = \max[0,(1+u)(1+d)S_{T-2} - K] = \max[0,(1+u)S_{T-1,d} - K] \\ C_{T,dd} = \max[0,(1+d)^2 S_{T-2} - K] = \max[0,(1+d)S_{T-1,d} - K] \end{cases}$$

Again, once we know that there was a downtick in the first period, we are in a single period world, and we can use the general equation, (17.8), to find the value of the call:

$$C_{T-1,d} = \frac{pC_{T,ud} + (1-p)C_{T,dd}}{1+r} \tag{17.10}$$

At this stage, we know the value of the call if there is an uptick in the first period, and we also know the value of the call if there is a downtick in the first period. In the recursive process, we now move back to time $T-2$, and again apply the single period BOPM principles:

$$S_{T-2} \begin{cases} S_{T-1,u} = (1+u)S_{T-2} \begin{cases} \cdots \\ \cdots \end{cases} \\ S_{T-1,d} = (1+d)S_{T-2} \begin{cases} \cdots \\ \cdots \end{cases} \end{cases}$$

$$C_{T-2} \begin{cases} C_{T-1,u} \begin{cases} \cdots \\ \cdots \end{cases} \\ C_{T-1,d} \begin{cases} \cdots \\ \cdots \end{cases} \end{cases}$$

At time $T-2$, we can invest in a levered portfolio that will provide the same payoffs at time $T-1$ as the call. The initial investment of $\Delta S_{T-2}+B$ will take on one of two values, depending on whether there is an uptick or a downtick during the first period:

$$\Delta S_{T-2} + B \begin{cases} (1+u)\,\Delta S_{T-2} + (1+r)B = \Delta S_{T-1,u} + (1+r)B \\ (1+d)\,\Delta S_{T-2} + (1+r)B = \Delta S_{T-1,d} + (1+r)B \end{cases}$$

Now, equate the time $T-1$ payoffs of the equivalent portfolio with the values of the call at time $T-1$:

$$\Delta(1+u)S_{T-2} + (1+r)B = C_{T-1,u}$$

$$\Delta(1+d)S_{T-2} + (1+r)B = C_{T-1,d}$$

These consist of two simultaneous equations with two unknowns, Δ and B. Solving them, we get:

$$\Delta = \frac{C_{T-1,u} - C_{T-1,d}}{(u-d)S_{T-2}} = \frac{C_{T-1,u} - C_{T-1,d}}{S_{T-1,u} - S_{T-1,d}} \tag{17.11}$$

$$B = \frac{(1+u)C_{T-1,d} - (1+d)C_{T-1,u}}{(u-d)(1+r)} \tag{17.12}$$

Note that the equations are identical to Equations (17.3) and (17.4). These equations define the investment in Δ shares of stock and bonds (but remember that for calls, $B \leq 0$) at time $T-2$, that creates a portfolio having payoffs that are the same as those from the call at time $T-1$.

If two assets or portfolios offer the exact same payoffs at time $T-1$, they must sell for the same price at time $T-2$. We have

$$C_{T-2} = \Delta S_{T-2} + B \tag{17.13}$$

where Δ and B are defined as in Equations (17.11) and (17.12). Now, substitute Equations (17.9) and (17.10) into (17.11) and (17.12), and then substitute the resulting expressions into Equation (17.13). The result is the value of a call when there are two periods before its expiration date:

$$C_{T-2} = \frac{p^2 C_{T,uu} + 2p(1-p)C_{T,ud} + (1-p)^2 C_{T,dd}}{(1+r)^2} \tag{17.14}$$

All the variables on the right-hand side of Equation (17.14) are known. The value of p is $(r-d)/(u-d)$. The time T value of the call can take on one of three values:

$$C_{T,uu} = \max[0, S_{T,uu} - K] = \max[0, (1+u)^2 S_{T-2} - K]$$

$$C_{T,ud} = \max[0, S_{T,ud} - K] = \max[0, (1+u)(1+d)S_{T-2} - K]$$

$$C_{T,dd} = \max[0, S_{T,dd} - K] = \max[0, (1+d)^2 S_{T-2} - K]$$

There is one more important concept to be learned in this theory. A portfolio of stocks and bonds can offer the same payoffs as any option. In other words, any option can be replicated by using stocks and bonds. However, the replication is a dynamic process. The values of Δ and B that we compute at time $T-2$ will not be the same as the values that exist at time $T-1$. Thus, at the end of every period, the composition of the equivalent portfolio changes. When you buy an option, there is a one-time cash outflow, and no other cash flow takes place until the option is sold or expires. If we are to claim that a dynamically adjusted portfolio of stocks and bonds is equivalent to an option, it must be the case that no other cash flows occur at all the equivalent portfolio's intermediate dates, when Δ and B change.

It turns out that the equivalent portfolio offers this property, which is called **self-financing**. Although Δ and B change over time, there are no additional cash requirements needed to continue the replicating process. If Δ increases, more shares must be bought at the new, higher stock price. However, there will be an increase in B equal to the product of the change in Δ and the new stock price. The value of the new shares to be purchased equals the increase in required borrowing.

Likewise, if the price of the underlying asset declines, Δ will decline, and shares will have to be sold to continue the replicating process. However, the proceeds from the sale of shares, at the new stock price, will exactly equal the decline in borrowing needed to maintain an equivalent portfolio. The self-financing property is illustrated in the example developed in Section 17.2.5.

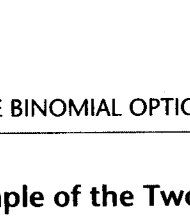
17.2.5 A Numerical Example of the Two-Period Binomial Option Pricing Model

EXAMPLE 17.1 The current stock price be \$20 per share. The stock can either rise by 20% per period or fall by 10% per period. The riskless interest rate is 10% per period. Find the value of a call option with a strike price of 20, having two periods until expiration. Also, at times $T-2$ and $T-1$, find the composition of the portfolio of stocks and riskless debt that is equivalent to the call, and demonstrate that the payoffs of this equivalent portfolio are indeed the same as the payoffs of the call. Finally, show that the equivalent portfolio is self-financing.

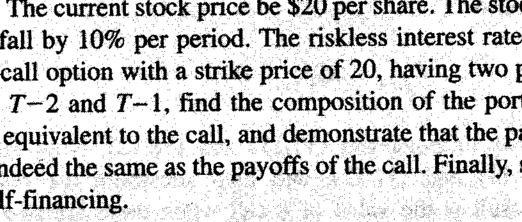
Solution Since u=0.2, d=–0.1, and r=0.1, the value of p is

$$p = \frac{0.1-(-0.1)}{0.2-(-0.1)} = \frac{2}{3} = 0.66667$$

Next, map out the stock price process (sometimes, this is called the "binomial tree"):



The pricing process for the call is



From here, the problem is solved recursively, moving from right to left, using the single-period BOPM pricing model at each step. If there is an uptick during the first period, Equation (17.9) determines the price of the call:

$$C_{T-1,u} = \frac{pC_{T,uu} + (1-p)C_{T,ud}}{1+r} = \frac{(2/3)(8.8) + (1/3)(1.6)}{1.1} = 5.\overline{8181}\ldots$$

If there is a downtick during the first period, Equation (17.10) determines the value of the call:

$$C_{T-1,d} = \frac{(2/3)(1.6) + (1/3)(0)}{1.1} = 0.969697$$

To find the value of the call at time $T-2$, we can use one of two approaches. First, since we know the value of the call at time $T-1$ if there is an uptick in the first period, and we also know the value of the call at time $T-1$ if there is a downtick in the first period, we are in a single-period situation that looks like:

$$C_{T-2} \begin{cases} 5.8\overline{181} = C_{T-1,u} \\ 0.969697 = C_{T-1,d} \end{cases}$$

The general single-period BOPM equation (17.8) can be used to value the call at time $T-2$:

$$C_{T-2} = \frac{(2/3)(5.8\overline{181}) + (1/3)(0.969697)}{1.1} = 3.8200184$$

Alternatively, we could use Equation (17.14) directly to find C_{T-2}:

$$C_{T-2} = \frac{(2/3)^2(8.8) + 2(2/3)(1/3)(1.6) + (1/3)^2(0)}{1.1^2} = 3.8200184$$

Next, we will find the composition of the equivalent portfolio at each date, using general formulas (17.3) and (17.4). At time $T-2$, Equations (17.11) and (17.12) are the versions of (17.3) and (17.4) that are used:

$$\Delta = \frac{C_{T-1,u} - C_{T-1,d}}{S_{T-1,u} - S_{T-1,d}} = \frac{5.8\overline{181} - 0.969697}{24-18} = 0.8080808$$

$$B = \frac{(1+u)C_{T-1,d} - (1+d)C_{T-1,u}}{(u-d)(1+r)} = \frac{(1.2)(0.969697) - (0.9)(5.8\overline{181})}{(0.3)(1.1)} = -12.341598$$

This says that an investor who bought 0.8080808 share of stock at its time $T-2$ price of \$20/share, and borrowed (because of the negative sign) \$12.341598, will have created a levered portfolio that pays off the same as the call, regardless of whether there is an uptick or a downtick.

The cost of the levered portfolio is:

$$\Delta S + B = (0.8080808)(20) - 12.341598 = 3.820018$$

This is the same as the value of the call we just computed, except for a small rounding error. If the stock has an uptick in the first period, then the levered portfolio will be worth:

$$\Delta S_{T-1,u} + (1+r)B = (0.8080808)(24) - (1.1)(12.341598) = 5.818182$$

which is the same value as the call if the stock rises at time $T-1$. If the stock declined in the first period, the levered portfolio would be worth:

$$\Delta S_{T-1,d} + (1+r)B = (0.8080808)(18) - (1.1)(12.341598) = 0.969697$$

which, again, is the same as the payoff provided by the call. Thus, we have shown that the portfolio consisting of 0.8080808 share of stock, financed in part with \$12.341598 in borrowed funds, offers the same payoffs as the call. Also, since the call and the levered portfolio offer identical payoffs at time $T-1$, they must be valued the same at time $T-2$, and they are.

Next, we must determine the composition of the two possible equivalent portfolios at time $T-1$. We will also verify that each is self-financing, so that no additional cash flows occur when replicating the option. Because there are two possible states of the world at time $T-1$, we must find the equivalent portfolio given that there was an uptick in the first period, and then repeat the calculations given that there was a downtick in the first period.

If there was an uptick in the first period, then by Equations (17.3) and (17.4) we find the composition of the new equivalent portfolio:

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{8.8 - 1.6}{28.8 - 21.6} = 1$$

$$B = \frac{(1+u)C_d - (1+d)C_u}{(u-d)(1+r)} = \frac{(1.2)(1.6) - (0.9)(8.8)}{[0.2 - (-0.1)](1.1)} = -18.181818$$

What this means is that although the equivalent portfolio at time $T-2$ consisted of 0.8080808 share of stock and −\$12.341598 in bonds, if the stock had an uptick in the first period, the new equivalent portfolio consists of one share of stock and B=−\$18.181818.[9] The equivalent portfolio is self-financing, though, because an additional 0.1919192 share of stock must be purchased at \$24/share, for a total additional investment in stock of \$4.60606. The old indebtedness was \$12.341598. After one period of interest, the option replicator owes \$13.575758 (1.1 × 12.341598). Since the new B in the equivalent portfolio is −\$18.181818, this means that (\$18.181818−\$13.575758=) \$4.60606 must be additionally borrowed!

What if the stock had a downtick in the first period? Then the composition of the equivalent portfolio would be:

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{1.6 - 0}{21.6 - 16.2} = 0.296296$$

$$B = \frac{(1+u)C_d - (1+d)C_u}{(u-d)(1+r)} = \frac{(1.2)(0) - (0.9)(1.6)}{[0.2 - (-0.1)](1.1)} = -4.363636$$

To continue replicating the option, (0.8080808 − 0.296296 =) 0.5117848 share of stock must be sold, at the new price of \$18/share. This provides \$9.212126 (0.5117848 × 18).

However, the new replicating portfolio also requires the borrowing of less money. The loan established at time $T-2$ was \$12.341598. Including interest, the total loan at time $T-1$ is then \$13.575758. If the proceeds from the sale of stock are used to repay the loan, the remaining loan balance is (\$13.575758 − \$9.212126 =) \$4.3636316. Note that this equals the new $-B$ value for the replicating portfolio (except for a small rounding error).

Thus, we have shown that at each date and possible stock price, there is a portfolio of stocks and bonds that will offer the same payoffs as the call. The dollar value of this portfolio equals the price of the call, at each date and possible stock price. Finally, the equivalent portfolio was shown to be self-financing. The final time path solution to the call values and composition of the equivalent portfolio is shown in Figure 17.2.

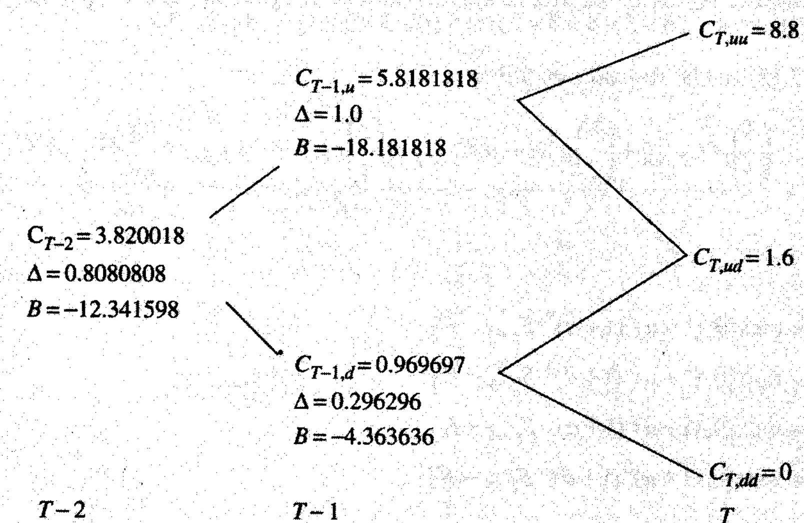


Figure 17.2 Call values and composition of the equivalent portfolio of Example 17.1. (Here and in other figures in this chapter, dollar signs are omitted for clarity.)

17.2.6 The Multiperiod Binomial Option Pricing Model

Equation (17.15), the three-period version of the BOPM, can be derived using the same logic as in Section 17.2.4:

$$C_{T-3}=\frac{p^3C_{T,uuu}+3p^2(1-p)C_{T,uud}+3p(1-p)^2C_{T,udd}+(1-p)^3C_{T,ddd}}{(1+r)^3} \tag{17.15}$$

We can rewrite this in a different way. First, we want to know how many ways a stock price can reach a terminal value in a binomial process. In other words, in n periods [in Equation (17.15), $n=3$], how many ways can the stock realize j upticks, where $j=0, 1, 2, 3$? Recall that if the stock pricing process is stationary over time, the order in which the upticks occur does not matter: *uud*, *udu*, and *duu* will all result in the same time T stock price. The answer is given by the binomial coefficient:

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

Thus, if $n=3$, how many ways can we get two upticks and one downtick? The answer is:

$$\frac{3!}{2!(3-2)!} = \frac{3 \times 2 \times 1}{(2 \times 1) \times (1)} = \frac{6}{2} = 3 \text{ ways}$$

How many ways can a stock realize 6 upticks in 10 periods? The answer is:

$$\frac{10!}{6!(10-6)!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210 \text{ ways}$$

Equation (17.15) can be rewritten as follows:

$$C_{T-3} = \frac{\binom{3}{3} p^3 C_{T,uuu} + \binom{3}{2} p^2 (1-p) C_{T,uud} + \binom{3}{1} p(1-p)^2 C_{T,udd} + \binom{3}{0} (1-p)^3 C_{T,ddd}}{(1+r)^3}$$

where

$$C_{t,uuu} = \max[0, (1+u)^3 (1+d)^0 \; S_{T-3} - K]$$

$$C_{t,uud} = \max[0, (1+u)^2 (1+d)^1 \; S_{T-3} - K]$$

$$C_{t,udd} = \max[0, (1+u)^1 (1+d)^2 \; S_{T-3} - K]$$

$$C_{t,ddd} = \max[0, (1+u)^0 (1+d)^3 \; S_{T-3} - K]$$

We can also write Equation (17.15) as follows:

$$C_{T-3} = \frac{1}{(1+r)^3} \sum_{j=0}^{3} \binom{3}{j} p^j (1-p)^{3-j} \max[0, (1+u)^j (1+d)^{3-j} S_{T-3} - K]$$

Thus, letting n equal the number of periods and j equal the number of upticks, the general multi-period BOPM for a call with n periods until expiration is:

$$C = \frac{1}{(1+r)^n} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max[0, (1+u)^j (1+d)^{n-j} S_{T-n} - K] \tag{17.16}$$

Now, let a be the minimum number of upticks needed to result in the call finishing in the money. This means that we will ignore all the outcomes in which the call's time T value is 0. If $j < a$, then the call is worthless. Then the model becomes:

$$C = \frac{1}{(1+r)^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} [(1+u)^j (1+d)^{n-j} S_{T-n} - K] \tag{17.17}$$

Note that $(1+u)^j(1+d)^{n-j}S_{T-n}=S_{T|j}$=the time T stock price, given that there are j upticks (and $n-j$ downticks). Furthermore, $(1+u)^j(1+d)^{n-j}S_{T-n}-K=C_{T|j}$=the time T call value, given that there are j upticks.

Equation (17.17) is a formidable-looking formula, but Example 17.2 will illustrate just how easy it is to use. Remember, this formula is to be used to value calls only when there are no ex-dividend days between today and the option's expiration date.

It is critical to learn the steps and logic behind the derivation of the multiperiod BOPM for two reasons. First, the model can be used to value calls on dividend-paying stocks, and in particular, to determine the times at which early exercise will be optimal for the call owner. And second, sometimes the user will believe that the pricing process behind the underlying asset will change over time (the u, d, and r need not be stationary over time). In both these cases, there is no simple model to determine the call's value. Rather, the recursive, step-by-step solution to the BOPM must be worked out. Fortunately, computers can do the job for us.

Cox and Rubinstein (1985) show that carving the time to expiration into as few as five intervals provides estimated call values reasonably close to those found by using the continuous time Black–Scholes option pricing model (BSOPM). However if great accuracy is important, and computation time (using a computer) is relatively unimportant, Cox and Rubinstein (1985) and Jarrow and Rudd (1983) recommend carving the time to expiration into 150 intervals. More intervals are needed when early exercise is a consideration.

17.2.7 A Numerical Example of the Multiperiod Binomial Option Pricing Model

EXAMPLE 17.2 Find the value of a call option on a non-dividend-paying stock that has the following parameters:

$n=6$ months

$u=0.05$/month

$d=-0.04$/month

$r=0.01$/month

$K=25$

$S_{T-6}=30$

This problem was developed when a class was asked: "Given that the common stock of XYZ sells for \$30/share, what is the highest reasonable stock price at which it might sell six months hence?" A typical answer was \$40–\$45 per share. The students were also asked to estimate the lowest stock price at which it could reasonably sell in six months. A typical answer was in the range of \$20–\$25 per share.

Solution When estimating u and d, it should always be the case that $u>|d|$, since stocks should always be priced to increase in value. They are risky assets, and in well-functioning markets ought to be priced to provide positive returns, almost always in excess of the riskless rate of interest. As you will see, we made the arbitrary selection of $u=0.05$ and

$d=-0.04$ to generate a highest price (given 6 upticks) and a lowest price (given 6 downticks) in the ranges believed possible. This is a reasonable approach to obtaining an estimate for u and d.

Table 17.1 presents the number of upticks (j), the number of ways that j upticks can be realized $\binom{n}{j}$, the resulting time T stock price (S_T), and the resulting expiration day call value ($C_T=S_T-K$).[10] Additionally, we can determine the value of p as follows:

$$p=\frac{r-d}{u-d}=\frac{0.01-(-0.04)}{0.05-(-0.04)}=0.55556$$

The parameter a is the minimum number of upticks needed to ensure that the call finishes in the money. When $K=25$, the value of a is 1. That is, if there is one uptick, then the call will finish in the money. If we were evaluating a call with a strike price of 30, then $a=3$.

TABLE 17.1 Computation of S_T and C_T in a Binomial Framework

j	$\binom{n}{j}$	S_T	C_T
6	$\frac{6!}{6!0!}=1$	$30(1.05)^6=40.2029$	15.2029
5	$\frac{6!}{5!1!}=6$	$30(1.05)^5(0.96)^1=36.7569$	11.7569
4	$\frac{6!}{4!2!}=15$	$30(1.05)^4(0.96)^2=33.6063$	8.6063
3	$\frac{6!}{3!3!}=20$	$30(1.05)^3(0.96)^3=30.7258$	5.7258
2	$\frac{6!}{2!4!}=15$	$30(1.05)^2(0.96)^4=28.0921$	3.0921
1	$\frac{6!}{1!5!}=6$	$30(1.05)^1(0.96)^5=25.6842$	0.6842
0	$\frac{6!}{0!6!}=1$	$30(1.05)^0(0.96)^6=23.4827$	0.0

Therefore, the multiperiod BOPM estimates the call's value to be:

$$C_{T-6} = \frac{1}{(1.01)^6}\left[\binom{6}{6}p^6(1-p)^0 C_{T|6} + \binom{6}{5}p^5(1-p)^1 C_{T|5} + \binom{6}{4}p^4(1-p)^2 C_{T|4} + \cdots\right]$$

which is:

$$C_{T-6} = \frac{1}{1.01^6}\begin{bmatrix} (1)(0.556)^6(0.444)^0(15.2029) + (6)(0.556)^5(0.444)^1(11.7569) \\ +(15)(0.556)^4(0.444)^2(8.6063) + (20)(0.556)^3(0.444)^3(5.7258) \\ +(15)(0.556)^2(0.444)^4(3.0921) + (6)(0.556)^1(0.444)^5(0.6842) + 0 \end{bmatrix}$$

$$= \$6.4599$$

17.3 Using the Binomial Option Pricing Model To Value Calls on Dividend-Paying Stocks

17.3.1 European Calls on Stocks That Pay a Discrete Percentage Dividend

The BOPM formula [Equation (17.17)] can be used to value European calls or American calls on non-dividend paying stocks. In this section, the equation is modified to treat the case in which the underlying stock pays a percentage of its value out in dividends at the end of one (or more) periods before the expiration date of the European call option. Here this percentage is assumed to be constant.

Saying that a stock will pay out a constant fraction of its value in dividends is just another way of saying that the stock has a constant dividend yield. For example, if the dividend yield is 1% and if the stock price is \$10/share, the dividend amount is \$0.10, and we assume that the stock will fall in value by that amount whenever it trades ex-dividend. Stocks do not pay dividends in this way. Firms almost always pay a fixed dollar amount, occasionally increasing that dollar amount when earnings increase.[11] But if foreign exchange, a well-diversified stock index, or a futures contract[12] is the underlying asset, the constant dividend yield model may be applicable.

Define δ as the constant dividend yield of the stock and assume that the dollar dividend amount ($= \delta S$) is paid at the end of one or more periods before the call's expiration date. Further assume that the stock price declines by the dividend amount when paid. The stock's ex-dividend date and dividend payment date are assumed to be the same. If there are m ex-dividend dates before expiration, n periods to expiration, and j upticks, then the time T stock price is:

$$S_T = (1+u)^j(1+d)^{n-j}(1-\delta)^m S_{T-n}$$

Suppose there are $n=5$ periods until expiration and that the stock will trade ex-dividend at the end of the first and fourth periods. Let the stock price today equal $S_{T-5}=20$, $u=0.1$, $d=-0.05$, and $\delta=0.02$. Then, Figure 17.3 shows the time path of possible stock prices.

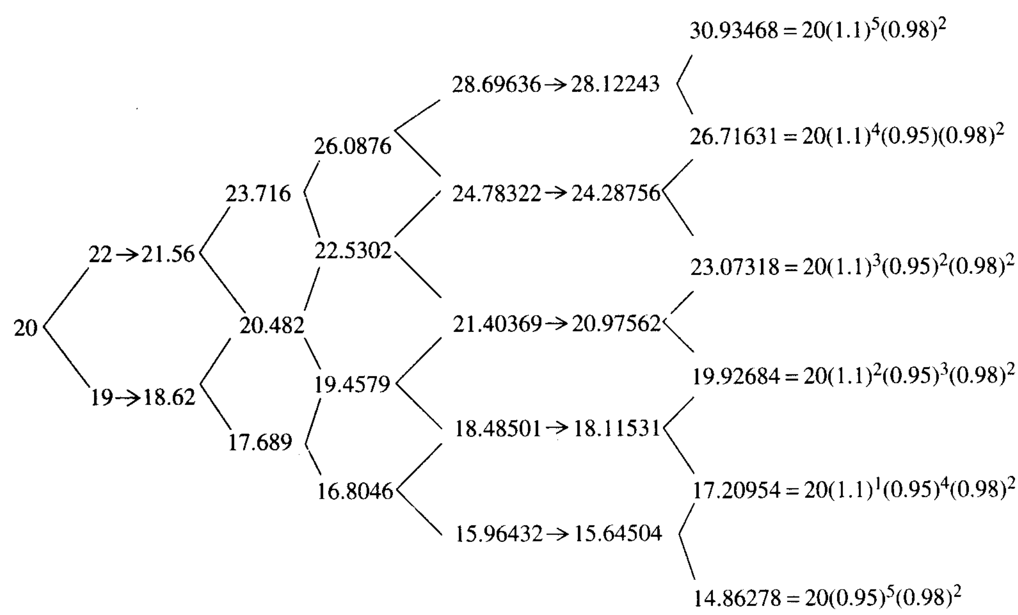


Figure 17.3

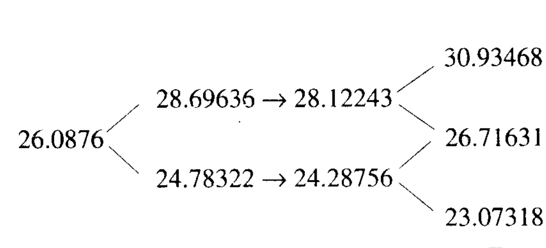
The arrows (→) denote that the stock trades ex-dividend. The stock never actually trades at the cum-dividend price. For example, in the first period, the stock either rises from 20 to 21.56 or falls from 20 to 18.62.

Note that in this constant dividend yield model, it does not matter *when* the ex-dividend days occur. We have specified that they occur at times $T-4$ and $T-1$, but the same time T stock prices would result if we specified *any* two dates.

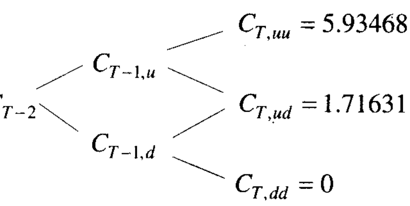
Since the options are European, the expiration day value of the call, given j upticks, is $\max[0, (1+u)^j(1+d)^{n-j}(1-\delta)^m S_{T-5} - K]$. The multiperiod BOPM is:

$$C = \frac{1}{(1+r)^n} \sum_{j=a}^{n} \binom{n}{j} [p^j (1-p)^{n-j}][(1+u)^j (1+d)^{n-j} (1-\delta)^m S_{T-n} - K] \tag{17.18}$$

The single-step, recursive process that leads to Equation (17.18) is the same as the one used in the no-dividend model. Let us just examine the upper right portion of the foregoing pricing process for the stock:



We will find the value of a call with a strike price of 25 at time $T-2$. At time $T-2$, $S_{T-2} = 26.0876$. Since $C_T = \max[0, S_T - K]$, we have the following call pricing process:



If we assume that $r=0\%$ per period, the value of p is:

$$p = \frac{r-d}{u-d} = \frac{0.00-(-0.05)}{0.10-(-0.05)} = 0.33333$$

Equation (17.18) can be used to find C_{T-2} in one easy step. Expanded, Equation (17.18) says:

$$\begin{aligned} C_{T-2} = \frac{1}{(1+r)^2} \{ & [1][p^2(1-p)^0][(1+u)^2(1+d)^0(1-\delta)^1 S_{T-2} - K] \\ & + [2][p^1(1-p)^1][(1+u)^1(1+d)^1(1-\delta)^1 S_{T-2} - K] \\ & + [1][p^0(1-p)^2][(1+u)^0(1+d)^2(1-\delta)^1 S_{T-2} - K] \} \end{aligned}$$

There must be at least one uptick for the call to finish in the money. Thus, $a = 1$, and the last term in brackets, for $j=0$ upticks, equals zero. Substituting values in for the symbols, we have:

$$\begin{aligned} C_{T-2} = \frac{1}{(1+0)^2} \{ & [1][(0.3333)^2(0.6667)^0][(1.1)^2(0.95)^0(0.98)^1(26.0876) - 25] \\ & + [2][(0.3333)^1(0.6667)^1][(1.1)^1(0.95)^1(0.98)^1(26.0876) - 25] + [0] \} \end{aligned}$$

$$C_{T-2} = 1\{[1][0.1111][30.93468 - 25] + [2][0.22222][26.71631 - 25]\} = 1.4222131$$

Alternatively, we could use the recursive method to find call values at each date and state of the world. First, use general equation (17.8) to compute $C_{T-1,u}$, and $C_{T-1,d}$.

$$C_{T-1,u} = \frac{(0.33333)(5.93468) + (0.66667)(1.71631)}{1.0} = 3.1224333$$

$$C_{T-1,d} = \frac{(0.33333)(1.71631) + (0.66667)(0)}{1.0} = 0.5721033$$

Then compute C_{T-2}:

$$C_{T-2} = \frac{(0.33333)(3.1224333) + (0.66667)(0.5721033)}{1.0} = 1.4222133$$

The compositions of the equivalent portfolios are found by using general equations (17.3) and (17.4). At time $T-2$, the call is in the following situation:

$$C_{T-2} \begin{cases} C_{T-1,u} = 3.1224333 \\ C_{T-1,d} = 0.5721033 \end{cases}$$

The equivalent portfolio will follow the following pricing process[13]:

$$\Delta S_{T-2} + B \begin{cases} \Delta S_{\text{ex},T-1,u} + \Delta \text{div}_u + (1+r)B = \Delta S_{\text{with-div},T-1,u} + (1+r)B \\ \Delta S_{\text{ex},T-1,d} + \Delta \text{div}_d + (1+r)B = \Delta S_{\text{with-div},T-1,d} + (1+r)B \end{cases}$$

Equate the payoffs of the equivalent portfolio to those of the call:

$$\Delta S_{\text{with-div},T-1,u} + (1+r)B = C_{T-1,u}$$
$$\Delta S_{\text{with-div},T-1,d} + (1+r)B = C_{T-1,d}$$

Compare these equations to those that led to general equations (17.3) and (17.4). They are identical, except that here, the with-dividend stock prices are used. This will always be the case if we assume that the stock price is expected to decline by the dividend amount. Consequently, when the two simultaneous equations in ΔS are solved, the solution for Δ uses the *with-dividend* stock price in the denominator:

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{3.122433 - 0.5721033}{28.69636 - 24.78322} = 0.6517349$$

Equation (17.4) is used to find the value of B at time $T-2$:

$$B = \frac{(1+u)C_d - (1+d)C_u}{(u-d)(1+r)} = \frac{(1.1)(0.5721033) - (0.95)(3.1224333)}{[0.1-(-0.05)](1)} = -15.57999$$

Thus, if you buy 0.6517349 share of stock at the $T-2$ price of 26.0876 and borrow \$15.57999 (at zero percent interest), the payoffs are equal to the call. If the stock price rises, the dividend is \$0.57393/share. Because you own 0.6517349 share, you get a dividend of \$0.37405. Add this to the new ex-dividend value of your shares, which is $(0.6517349 \times 28.12243 =)\$18.3282369$, and you have \$18.70242. Repay the loan of \$15.57999, and the time $T-1$ ex-dividend value of the equivalent portfolio is \$3.12243. Except for a miniscule rounding error, this equals $C_{T-1,u}$.

Given there is an uptick in the first period, at time $T-1$, the equivalent portfolio's composition is:

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{5.93468 - 1.7631}{30.93468 - 26.71631} = 1.0$$

$$B = \frac{(1+u)C_d - (1+d)C_u}{(u-d)(1+r)} = \frac{(1.1)(1.71631) - (0.95)(5.93468)}{[0.1-(-0.05)](1)} = -25.0$$

The payoffs of the equivalent portfolio are identical to the call's payoffs. If there is an uptick in the second period, the value of the single share rises to \$30.93468. Subtract the (interest-free) loan of \$25.0, and the equivalent portfolio is worth \$5.93468. The latter equals $C_{T,uu}$.

Furthermore, the equivalent portfolio is self-financing if there is an uptick in the first period. At time $T-2$, you had 0.6517349 share. You also received a dividend of \$0.37405, which can be used to buy 0.0133008 additional share of stock at the time $T-1$ ex-dividend price of \$28.12243/share. Thus, you have 0.6650357 share, ex-dividend, and the new equivalent portfolio requires that you have 1.0 share. Therefore, you must spend \$9.420011 to buy the additional 0.3349643 share of stock at \$28.12243/share. However, you originally had borrowed \$15.57999 at an interest rate of zero, but now the equivalent portfolio requires you to borrow \$25. The additional borrowing of \$9.42001 will exactly pay for your additional shares of stock (except for a small rounding error).

Finally, the same analysis can be applied if there is a downtick in the first period. This is left as an exercise for the student (see Problem 17.8 at the end of the chapter).

In summary, if we assume the stock has a constant dividend yield, the multiperiod BOPM is easily modified to value European calls. You must, however, know the number of times the stock will trade ex-dividend before option expiration. Then, the possible time T stock prices are

$$S_T = (1+u)^j (1+d)^{n-j} (1-\delta)^m S_{T-n}$$

when there are n periods to expiration and m ex-dividend dates prior to that date. Just let $j=0, 1, \ldots, n$ upticks to find each possible stock price at expiration. The formulas to find the equivalent portfolio are the same as we found earlier, except that if the stock trades ex-dividend at the next date, you must remember to use the with-dividend stock prices to find Δ.

17.3.2 European Calls on Stocks Paying a Discrete, Known Dividend Amount

Here, we assume that we know the dates at which the stock will trade ex-dividend, paying a known dollar dividend amount. The model can even be extended to assume that if the stock has an uptick, there will be a known dividend amount, div_u, and if the stock has a downtick, it will pay a different known amount, div_d. This provides additional realism to the stock pricing process being modeled, since if the stock rises, it is likely that the company's profits have risen and a higher dividend will be paid. One of the great advantages of the BOPM is that such situations can be easily handled. All you need to understand is the basic single-period BOPM logic.

Unlike the constant dividend yield BOPM, there is no "simple" equation, such as Equation (17.18), that provides a call value. This is because when the stock pays a dividend *amount*, it becomes important to know the exact ex-dividend dates. In the constant dividend yield example of Section 17.3.1, it mattered only that the stock was going to trade ex-dividend twice before the expiration date. Now, it matters *when* those dividends will be paid. For example, consider Figures 17.4 and 17.5, in which $n=3$, $u=0.1$, and $d=-0.05$. In Figure 17.4 the stock trades ex-dividend in the amount of \$1 at time $T-2$. In Figure 17.5, the stock trades ex-dividend in the amount of \$1 at time $T-1$. As you can see, the time T stock prices differ. Also, the number of possible time T stock prices differ depending on when the ex-dividend day occurs. The impact of this is that to use the BOPM to value a European call, it is necessary to use the single-period model recursively.

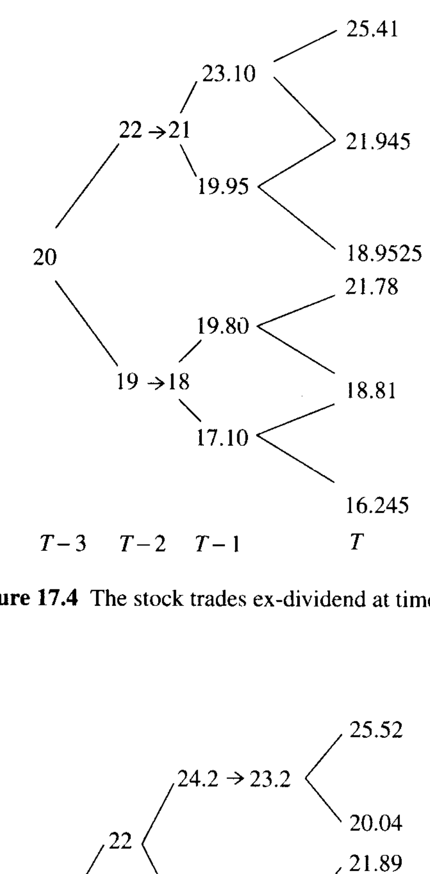


Figure 17.4 The stock trades ex-dividend at time $T-2$.

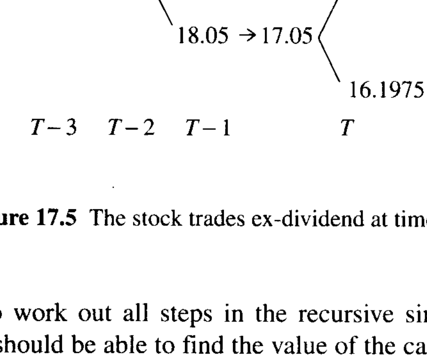


Figure 17.5 The stock trades ex-dividend at time $T-1$.

You should be able to work out all steps in the recursive single-period BOPM. For each outcome at each date, you should be able to find the value of the call, find the composition of the equivalent portfolio, verify that the payoffs of the equivalent portfolio and the call are equal, and verify that the equivalent portfolio is self-financing.

17.3.3 American Calls on Stocks Paying a Given Dollar Dividend Amount

The recursive single-period approach must be used when American calls on stocks paying a given dollar dividend are to be valued by means of BOPM. One benefit of the model is that the dates and outcomes in which the call will be exercised early will be specified. In fact, you can even estimate the probability of early exercise.

At each date and state of the world, the value of the American call is:

$$\max\left[\frac{pC_u + (1-p)C_d}{1+r}, S-K\right] \tag{17.19}$$

In other words, you first use Equation (17.7) to find the value of the call. If the resulting value is less than the intrinsic value, Proposition II′ from Chapter 16 has been violated. Therefore, in these cases, the American call will sell for its intrinsic value instead of the value from Equation (17.7).

For any outcome at any date before expiration, the American call will be exercised early by rational investors whenever it sells for $S-K$. This is because investors can use the equivalent portfolio of stocks and bonds to replicate the American call's payoffs at a lower cost than the price of the call (its intrinsic value). In other words, an investor who wished to own the call eould replicate it at the lower cost of $[pC_u+(1-p)C_d]/(1+r)$, rather than buy the call for $S-K$. Likewise, if an investor currently owned the call, and wished to continue owning it, it would be rational to exercise it and then replicate it using stocks and bonds. Again, the cost of replication is less than $S-K$.

To illustrate, consider the example in Figure 17.5. We will value both an American call and a European call; the strike price is 20. First, use Equation (17.7) to value the call at each date and state of the world. We work from right to left. The values of u, d, r, and p are 0.1, –0.05, 0.0, and 1/3, respectively.

Time $T-1$

$C_{T-1,uu}$ → $C_{T,uuu} = 5.52$, $C_{T,uud} = 2.04$

$$C_{T-1,uu} = \frac{pC_{T,uuu} + (1-p)C_{T,uud}}{1+r} = 3.20$$

$C_{T-1,ud}$ → $C_{T,udu} = 1.89$, $C_{T,udd} = 0.0$

$$C_{T-1,ud} = \frac{pC_{T,udu} + (1-p)C_{T,udd}}{1+r} = 0.63$$

$C_{T-1,dd}$ → $C_{T,ddu} = 0.0$, $C_{T,udd} = 0.0$

$$C_{T-1,dd} = \frac{pC_{T,ddu} + (1-p)C_{T,ddd}}{1+r} = 0.0$$

Time $T-2$

$$C_{T-2,u} \quad \begin{array}{l} \nearrow C_{T-1,uu} = 3.20 \\ \searrow C_{T-1,ud} = 0.63 \end{array} \qquad C_{T-2,u} = \frac{pC_{T-1,uu} + (1-p)C_{T-1,ud}}{1+r} = 1.486667 \text{ (European)}$$

Because this value is less than the call's intrinsic value, the American call sells for $S_{T-2,u} - K$, which is 2. At this price, all rational owners of the call will exercise it, because the same call can be replicated with an equivalent portfolio of stocks and bonds at a cost of only $1.486667 (which is the value of the European call). The American call owner can exercise the call, paying $20 for a stock that he then sells for $22, resulting in a cash inflow of $2. Then, he can invest in a levered portfolio that offers the same payoffs as the call but requires a cash outlay of only $1.486667. The final result is a cash inflow of $0.51333 at time $T-2$. However, the cash flows at time $T-1$ are the same regardless of whether the investor owns the call or invests in the levered portfolio.

Continuing with the call value computations, we write:

$$C_{T-2,d} \quad \begin{array}{l} \nearrow C_{T-1,du} = 0.63 \\ \searrow C_{T-1,dd} = 0.00 \end{array} \qquad C_{T-2,d} = \frac{pC_{T-1,du} + (1-p)C_{T-1,dd}}{1+r} = 0.21$$

There are two values (one American, and one European) for the call to be determined at time $T-3$. The American call is valued by using the American call values at time $T-2$:

$$C_{T-3} \quad \begin{array}{l} \nearrow C_{T-2,u} = 2 \\ \searrow C_{T-2,d} = 0.21 \end{array} \qquad C_{T-3} = \frac{pC_{T-2,u} + (1-p)C_{T-2,d}}{1+r} = 0.806667 \text{ (American)}$$

Because 0.806667 is greater than $S-K$, the American call will not be exercised early at time $T-3$. The value of the European call at time $T-3$ utilizes the European call values at time $T-2$.

$$C_{T-3} \quad \begin{array}{l} \nearrow C_{T-2,u} = 1.486667 \\ \searrow C_{T-2,d} = 0.21 \end{array} \qquad C_{T-3} = \frac{pC_{T-2,u} + (1-p)C_{T-2,d}}{1+r} = 0.635556 \text{ (European)}$$

Next, find the equivalent portfolios at each date and possible outcome. In each case, we use Equations (17.3) and (17.4).

$$\Delta = \frac{C_u - C_d}{S_u - S_d} \tag{17.3}$$

$$B = \frac{(1+u)C_d - (1+d)C_u}{(u-d)(1+r)} \tag{17.4}$$